Mathematical Investigation of Solitary Waves in Deep and Shallow Waters

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ABSTRACT

Soliton waves are waves that preserve their well-defined shape when collide with each other. The observation of solitary waves dates back to 1834 by J. Scott Russell, who discovered the existence of solitary waves and also their speed. In 1845, Airy presents a formula shows the relation between the speed and the height and amplitude of the wave. Airy concluded that a solitary waves could not exist. Later, Korteweg and de Vries proved mathematically the existence of solitary waves; this was in the derivation of their KdV nonlinear Partial Differential Equation. Since then a lot of numerical work have been made to solve the KdV equation. The first work attempt to obtain an exact solution to the KdV, was that of Ryogo Hirota in 1971. The aim of this thesis, is to investigate solitary waves in deep and shallow waters via Hirota direct method. We consider two types of soliton waves on deep and shallow waters described respectively by Boussinesq and shallow waters equations. We show that the waves on shallow water have two main properties the: small wavelength and slower speed, in opposite to the waves on deep water are always faster with large wavelength. We use a computer programming (Mathematica) to show how solitary waves with long height come across shorter waves and pass them without any effect.
استقصاء رياضي للموجات المنعزلة للموجات المنعزلة في المياه العميقة والضحلة

محمد عثمان سليمان عثمان

ملخص الدراسة

الموجات المنعزلة تحتفظ بشكلها عندما تصطدم مع بعضها البعض. تم التعرف على الموجات المنعزلة منذ زمن بعيد يرجع إلى عام 1834 عن طريق العالم إسكتي آريان الذي اكتشف وجود الموجات المنعزلة وحدد مقدار سرعتها. لاحقا في عام 1845 وضع إيري صيغته توضح علاقة السرعة بطول وسعة الموجة. إيري توصل لخلاصه أن الموجات المنعزلة غير موجودة بيد أنه في عام 1895 أظهر العالمان كارتوك ودي فرز رياضيا وجود الموجات المنعزلة من خلال اشتقاق المعادلة التفاضلية الجزئية غير الخطية (KdV). ايرجي توصل لخلاصه أن الموجات المنعزلة غير موجودة. مقدماً عام 1895. مقدماً (KdV). أول عمل تم فيه الوصول لحل دقيق عن طريق العالم هيريرا. الهدف من هذا البحث هو الوصول إلى اشكال الموجات المنعزلة في المياه الضحلة والعميقة باستخدام طريقة هيريرا المباشرة. اخذنا في الاعتبار نوعين من الموجات المنعزلة في المياه الضحلة والعميقة التي توصف بمداحة المياه الضحلة ومعادلة بوزنتست على الترتيب. اوضحنا أن الموجات في المياه الضحلة لها خاصيتين أساسيتين قصيره وبطنية على العكس بالنسبة للموجات في المياه العميقة فانها دائما سريعة وذات اتساع أكبر. استخدمنا برمجة الحاسوب لتوضيح كيف أن الموجات المنعزلة ذات الارتفاع العالي تعتبر خلال الموجات ذات الارتفاع المنخفض وتمر دون تأثير عليها وبالتالي على حسب علماً وصولنا إلى هذه الحقائق يعتبر من الشيء الجديد.
I express my deep thanks and gratitude to Allah for giving me the ability to complete this work. I am also grateful to my supervisor Dr. Mogtaba Ahmed Yousif for his continuous guidance and valuable suggestions and comments at the different stages of the study. Many thanks and appreciation are due to my co-supervisor Dr. Faiz Awad Special thanks go to my family for their continuous encouragement to complete my work.
DEDICATION

This Thesis is dedicated to

My Father

My Mother

My Brothers and Sisters
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CHAPTER ONE
INTRODUCTION

1.1 Introduction

Soliton waves preserve their well-defined shape when collide with each other. The observation of solitary waves dates back to 1834 by J. Scott Russell, discovers the existence of solitary waves and also their speed. Some mathematicians were doubting that, his results may not be correct. In 1845, Airy present a formula shows the relation of speed with the height and amplitude of the wave. Airy conclusion was that a solitary waves could not exist. Later, Korteweg and de Vries proved mathematically the derivation of their KdV equation. Since then a lot of numerical work have been made to solve the KdV equation. The first work attempt to obtain an exact solution to the KdV was that of Ryogo Hirota in, (Hirota, 1971). His ingenious method was used to solve the nonlinear problem, [ (Hirota, 1971) (Hirota, 1972)]. The main ingredients of Hirota method could be listed in the following steps:

Step 1. To make suitable transformations of nonlinear PDEs to quadratic form, this is called Hirota bilinear form.

Step 2. To write the bilinear form in step 1 as polynomial in an operator D called the D-operator.

Step 3. To use perturbation expansion of the solution in the Hirota bilinear form obtained in steps 1 and 2. After that, analyse the perturbation parameter and its powers. From the last step one can be able to obtain N-soliton of the nonlinear PDEs, with condition that the given PDE is integrable. As mentioned above the speed of the wave related to the height and amplitude of the solitary wave. In this work, we consider solitary waves in deep and shallow waters and compare between the two types of these waves.
1.2 PROBLEM STATEMENT

One of the most challenging nonlinear phenomena is the understanding of the behavior of nonlinear waves on oceans. In this study, we will give some answers about questions in this direction by considering some nonlinear partial differential equations. The questions are of the following types:

- What is the speed of waves in terms of its height and its width?
- What will happen when two solitary waves cross each others?
- What is the difference between the solitary waves on deep and shallow waters?

1.3 RESEARCH OBJECTIVES

The aim of this study is to use Hirota’s method to obtain soliton solutions to nonlinear partial differential equations and to investigate and understand different properties of solitary waves on deep and shallow waters. Therefore our objectives are as follows:

- To solve analytically nonlinear partial differential partial differential equations via Hirota’s direct method.
- To use computer programming in order to understand and explain the behaviour of the nonlinear waves, in particular the solitary waves on deep and shallow waters.
• Also to see what will happen when the faster waves collide with the slower one

CHAPTER TWO
METHODOLOGY

In this chapter, we present in details the methodology used this study. We start by giving the notions and definitions the will be great importance in our analysis.

2.1 MAIN CONCEPTS AND DEFINITIONS

Definition 2.1

Let \( S = [f = \mathbb{C}^n \rightarrow \mathbb{C}^n] \) be the set of all differentiable functions, we define the D-operator

\[
D = S \times S \rightarrow S , \text{ by:}
\]

\[
[D^{m_1}_x D^{m_2}_t ...][f, g] = [(\partial_x - \partial_{\hat{x}})^{m_1}(\partial_t - \partial_{\hat{t}})^{m_2}][f(x, t, ...)g(x, t, ...)|x = \hat{x}, t = \hat{t}
\]

where \( m_i, i = 1,2,3... \) are positive integers and \( x, t, ... \) are independent variables. This operator is called Hirota operator.

Some of the properties of the D-operator are listed below:

(1) \( D_x (f \cdot g) = f_x g - fg_x \)

(2) \( D_x^2 (f \cdot g) = f_{xx} g - 2f_x g_x + fg_{xx} \)

(3) \( D_x^3 (f \cdot g) = f_{xxx} g - 3f_{xx} g_x + 3f_x g_{xx} - fg_{xxx} \)

(4) \( D_x^4 (f \cdot g) = f(x) = f_{xxxx} g - 4f_{xxx} g_x + 8f_{xx} g_{xx} - 4f_x g_{xxx} + fg_{xxxx} \)
Lemma 2.2

The following assertions are true

\[ \begin{align*}
(1) & \quad \frac{D_x^2(f.f)}{f^2} = u = 2 \frac{\partial^2}{\partial x^2} \log f(x, t) \\
(2) & \quad \frac{D_x D_t(f.f)}{f^2} = \ln(f^2)_{xt} \\
(3) & \quad \frac{D_x^2(f.f)}{f^2} = \iint u_{tt} \, dx \, dx \\
(4) & \quad \left( \frac{D_x^2(f.f)}{f^2} \right)^2 = \left( \frac{D_x^2 f.f}{f^2} \right)^2 - 3 \left( \frac{D_x f.f}{f^2} \right)^2
\end{align*} \]

Proof

\[ \begin{align*}
\left( \frac{D_x^2 f.f}{f^2} \right) &= u \\
\left( \frac{D_x^2 f.f}{f^2} \right) &= \frac{2 f_{xx} - 2 f_x^2}{f^2} \quad \rightarrow (i)
\end{align*} \]

\[ u = 2(\ln f(x, t))_{xx} = \frac{\partial^2 \ln f}{\partial x^2} \rightarrow \frac{\partial}{\partial x} \left( \frac{2 f x}{f} \right) = \frac{2 f_{xx} f - 2(f_x)^2}{f^2} \quad \rightarrow (ii) \]

\[ (i) = (ii) \]

Proof of (2)

\[ ((\ln(f^2))_x)_t = \left( 2 \frac{f_x}{f} \right)_t \rightarrow \frac{2 f_{xt} f - 2 f_x f_t}{f^2} \quad \rightarrow (i) \]
Proof of (3)

\[ u = 2(\ln f(x, t))_{xx} \]

\[ u_{tt} = [2(\ln f(x, t))_{xx}]_{tt} \]

\[ = [2(\ln f(x, t))_{tt}]_{xx} \]

\[ = \frac{2f_{xx}f - 2f_x}{f^2} \rightarrow \]

\[ \frac{D^2(f, f)}{f^2} = \frac{2f_x f_t - 2(f_t)^2}{f^2} \rightarrow \]

\[ \iint u_{tt} \, dx = \frac{2f_x f_t - 2(f_t)^2}{f^2} = \frac{D^2(f, f)}{f^2} \]

**Proposition 2.3**

Let \( p(D) \) a differentiable operator acting on two differentiable functions, then

\[ p(D)(f, g) = p[-D] [g, f] \]

Proof

Take \( p(D) = D_x^m \). The other combinations of \( D \)-operators follow in some manner we can write

\[ p(D)[f, g] = D_x^m[f, g] \]

\[ = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f_{(m-k)x} g_{kx} \]

\[ = f_{mx} g - m f_{(m-1)x} g_x + \cdots + (-1)^m f g_{mx} \]

\[ = (-1)^m \left[ f g_{mx} - m f_x g_{(m-1)x} + \cdots + (-1)^{m-1} m f_{(m-1)x} g_x + (-1)^m f_{mx} g \right] . \]
Proposition 2.4

Let \( p(D) \) acting on two exponential \( e^{\theta_1} \) and \( e^{\theta_2} \) where \( \theta_i = k_i x + \cdots + r_i z + l_i y + \alpha_i \),

and \( k_i, r_i, l_i, \alpha_i \) and constants for \( i = 1, 2 \). Then we have

\[
p(D)[e^{\theta_1} e^{\theta_2}] = p((k_1 - k_2), \ldots, (r_1 - r_2), (l_1 - l_2)) e^{\theta_1 + \theta_2}
\]

Proof

We consider the operator \( p(D) \) as:

\[
p(D) = [D_x^{m_1} \ldots D_z^{m_{n-1}} D_y^{m_n}], \quad \text{where } m_i, i = 1, 2, 3, \ldots, n,
\]

are positive integer and \( x, \ldots, y, z \) are independent variables when \( p(D) \) acts on the product of the exponential function \( e^{\theta_1} \) and \( e^{\theta_2} \) where \( \theta_i = k_i x + \cdots + r_i z + l_i y + \alpha_i \), we have

\[
p(D)[e^{\theta_1} e^{\theta_2}] = [D_x^{m_1} \ldots D_z^{m_{n-1}} D_y^{m_n}][e^{\theta_1} e^{\theta_2}]
\]

\[
= (l_1 - l_2)^{m_n}[D_x^{m_1} \ldots D_z^{m_{n-1}}][e^{\theta_1} e^{\theta_2}]
\]

\[
- (r_1 - r_2)^{m_{n-1}} (l_1 - l_2)^{m_n} [D_x^{m_1} \ldots D_z^{m_{n-2}}][e^{\theta_1} e^{\theta_2}]
\]

We continue keep this operating on the we exponential function, we will get:

\[
p(D)[e^{\theta_1} e^{\theta_2}] = (k_1 - k_2)^{m_1} \ldots (r_1 - r_2)^{m_{n-1}} (l_1 - l_2)^{m_n} (e^{\theta_1 + \theta_2}).
\]

In the sake of shortening the notations we shall use \( p(p_1 - p_2) \) instead of \( p(k_1 - k_2) \ldots (l_1 - l_2) \)

Corollary 2.4

Consider the operator \( p(D) \) and the functions \( f \) and \( g = 1, \) then:

\[
p(D)[f \cdot 1] = p(\partial)f, \quad p(D)[1 \cdot f] = p(-\partial)f
\]
Corollary 2.5

Let $a$ be a non-zero constant. Then of $p(D)[a,a] = 0$, we have $p(0,0,...,0) = 0$

Remark:

Due the symmetry of $p(D)$, the odd terms vanish, and only the even terms will remain, Thus we can say that $p(D)[f,f]$ is an even function. Furthermore, we have

$[D_x^{m_1} ... D_z^{m_{n-1}} D_y^{m_n}][f,f] = 0$, when $\Sigma_{i=1}^{k} m_i = odd$, for instance,

$D_x[f.f] = f_x f - f f_x = 0$

$D_t D_x^2 [f,f] = f_{xx} f_t - f_{xx} f_t - f_{xx} f_x + f_{xx} f_x + f_x f_{xx} + f_t f_{xx} - f_{xx} f_t = 0$.

2.2 Hirota’s Bilinear Equation:

Let us consider Hirota bilinear form:

$p(D)[f,f] = 0,$

(1.2.1)

Where $p(D)$ is polynomial in the Hirota D-operator this equation is bilinear in $f$.

The best example for equation (1.2.1) is the KdV equation

$u_t + 6uu_x + u_{xxx} = 0$

(1.2.2)

To bilinearize equation (1.2.2), let us

$u = 2 (\log f)_{xx}$

(1.2.3)

With the assumption $u = w_x$, then we have:
\[ w = 2 (\log f)_x \]  \hspace{1cm} (1.2.4)

\[ w_{xt} + 6w_x w_{xx} + w_{xxxx} = 0. \]  \hspace{1cm} (1.2.5)

Integrating with respect to \( x \) we yield the flowing expression

\[ w_t + 3w_x^2 + w_{xxx} = c \]  \hspace{1cm} (1.2.6)

using \( u = 2 (\log f)_{xx} \), in (1.2.5), we obtain

\[ 2\partial_t \partial_x (\log f) + 3(\partial_x^2 \log)^2 + 2\partial_x^4 (\log f) = c \]  \hspace{1cm} (1.2.7)

where \( c \) is the integration constant

applying Lemma 2.2 into (1.2.7) to get:

\[ \frac{D_x D_t [f, f]}{f^2} + 3 \left( \frac{(D_x^2 [f, f])}{f^2} \right)^2 + \frac{D_x^4 [f, f]}{f^2} - 3 \left( \frac{(D_x^2 [f, f])}{f^2} \right)^2 = c. \]

this gives

\[ D_x D_t [f, f] + D_x^4 [f, f] = cf^2 \]  \hspace{1cm} (1.2.8)

Letting \( c \equiv 0 \), we have

\[ D_x (D_t + D_x^2)(f, f) = 0. \]  \hspace{1cm} (1.2.9)

Equation (1.2.9) is the Hirota Bilinear form for the KdV. (Druitt, 2005)

As a second example, we consider the KP equation. That is:

\[ u_t - 6uu_x + (u_{xxx})_x + 3u_{yy} = 0 \]  \hspace{1cm} (1.2.10)

As before let:

\[ u(x, t, y) = -2\partial_x^2 \log f \]

The bilinear form of the KP is
\[ff_{xt} - f_x f_t + 3f^2 + ff_{xxxx} - 4f_x f_{xxx} + 3f_{yy} f - 3f_y^2 = 0 \]  \hspace{1cm} (1.2.11)

From Lemma2.2 and (1.2.11), we obtain the Hirota bilinear form of the KP equation

\[(D_x D_t + D_x^4 + 3D_x^2)(f \cdot f)\]

We see in the following example that, Hirota bilinear form may take more than of one equation

Example2.6

The modified Kdv (MKdV) has the following form:

\[u_t + 24u^2 u_x + u_{xxx} = 0 \]  \hspace{1cm} (1.2.12)

We use the following substitution for the bilinearizing transformation of MKdV equation

\[u(x,t) = \frac{g_x f - gf_x}{g^2 + f^2} \]  \hspace{1cm} (1.2.13)

We get:

\[-(g^2 + f^2)(g_t f - g f_t) + g_{xxx} f - 3g_{xx} f_x - gf_{xxx} + 6(f g_x - g f_x)(f f_{xx} - f_x^2 + gg_{xx} - g_x^2 = 0 \]  \hspace{1cm} (1.2.14)

From Lemma2.2 and (1.2.14), we obtain the following forms

\[(D_x^3 + 3D_t)(g, f) = 0 \]

\[(D_x^2)(f \cdot f + gg)\]

2.3 Hirota’s Method:

In this section, we present the Hirota method acting on a nonlinear partial differential equation. See for instance (Druitt · 2005).

\[F(u) = 0 \]  \hspace{1cm} (1.3.1)
We deduce that the bilinear form we obtain the exact solution at (1.3.1) for this let us write

\[ f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots, \]

where \( f_0 \) is constant and \( f_i, i = 1, 2, \ldots \) are function of \( x, t \) and \( c \) is const for \( f_0 = 1 \) we have the flowing

\[ f \cdot f = 1.1 + \epsilon (f_1 \cdot 1 + 1. f_1) + \epsilon^2 (f_2 \cdot 1 + f_1 \cdot f_1 + 1. f_2) + \epsilon^3 (f_3 \cdot 1 + f_2 \cdot f_2 + f_1 \cdot f_2 + 1. f_3) + \cdots \tag{1.3.2} \]

Substituted the above equation into the bilinear equation \( p(D)(f \cdot f) = 0 \) and collecting the coefficients for the power of \( \epsilon \) we obtain the flowing

\[ p(D)(f \cdot f) = p(D)(1.1) + \epsilon p(D)(f_1 \cdot 1 + 1. f_1) + \epsilon^2 p(D)(f_2 \cdot 1 + f_1 \cdot f_1 + 1. f_2) + \epsilon^3 p(D)(f_3 \cdot 1 + f_2 \cdot f_2 + f_1 \cdot f_2 + 1. f_3) + \cdots \tag{1.3.3} \]

\[ \epsilon = p(D)(f_1 \cdot 1 + 1. f_1) = 0 \]

\[ \epsilon^3 = p(D)(f_3 \cdot 1 + f_2 \cdot f_2 + f_1 \cdot f_2 + 1. f_3) = 0 \]

The solution of equation (1.3.3) may take several format one of these forms is the exponential which we will consider in our analysis. Let us assume that \( f \) is an exponential function and \( f_i.s \) to n.soliton solution of equation (1.3.1) we will consider

\[ f_i = 0 \ \forall i \geq n + 1, \ . \]

The following theorems give the conditions for finding one soliton solution

\textbf{Theorem 2.7}

Let \( u = T[f(x, t, \ldots, y)] \), be bilinearizing transformation of a nonlinear Partial differential equation \( F[u] = 0 \), which can be written in the Hirota bilinear form \( p(D)[f.f] = 0 \). Then one soliton solution of this equation is: \( u = T[f(x, t, \ldots, y)] = \)
$T[1 + e^{\theta_1}]$ where $\theta_1 = k_1 x + w_1 t + \cdots + l_1 y + \alpha_1$ with the constants $k_1, w_1, \ldots, l_1$ satisfying $p(k_1, w_1, \ldots, l_1) = p(p_1) = 0$.

Proof:

For finding one-soliton solution of

$$F[u] = 0,$$  \hfill (1.10.1)

we consider the following expansion

$$f = 1 + \epsilon f_1,$$  \hfill (1.10.2)

where

$$f_1 = e^{\theta_1},$$  \hfill (1.10.3)

with $\theta_1 = k_1 x + w_1 t + \cdots + l_1 y + \alpha_1$.

we have $f_i = 0$, for all $j \geq 2$. After inserting $f$ into the equation (1.3.3), we make the coefficients of $\epsilon^m, m = 0, 1, 2$ to be vanish.

The coefficient of $\epsilon^0$ is

$$p(D)[1.1] = p(0,0, \ldots, 0 \{1\}$$

and it vanishes trivially by corollary (1.6) makes the coefficient of $\epsilon^1$ turns out to be

$$= 2p(\partial)e^{\theta_1}$$

The above to zero and using the proposition (1.4) we obtain

$$p(k_1, w_1, \ldots, l_1) = p(p_1) = 0.$$  

This relation is called as the dispersion relation.

Since $f_2 = 0$, the coefficient of $\epsilon^2$ becomes
\[p(D)(f_2.1 + 1.f_2) + p(D)[f_1, f_1] = p(D)[e^{\theta_1} e^{\theta_1}] = p(p_1 - p_1)e^{2\theta_1}\]

and it is identically zero. Without loss of generality, we may set \( p = 1 \) . So \( f = 1 + e^{\theta_1} \) and one-soliton solution of \( F[u] = 0 \) is

\[u = T[f(x, t, ..., y)] = T[1 + e^{\theta_1}]\]

Where \( \theta_1 = k_1 x + w_1 t + \cdots + l_1 y + \alpha_1 \)

with the constants \( k_1, w_1, ..., l_1 \) satisfying

\[p(p_1) = 0.\]

**Theorem 2.8**

Let \( u = T[f(x, t, ..., y)] \) be bilinearizing transformation of a non-linear partial differential equation \( F[u] = 0 \), which can be written in the Hirota bilinear form \( (D)[f, f] = 0 \) . Then two-soliton solution of this equation is

\[u = T[f(x, t, ..., y)] = T[1 + e^{\theta_1} + e^{\theta_2} + A(1, 2)e^{\theta_1+\theta_2}]\]

Where \( \theta_1 = k_1 x + w_1 t + \cdots + l_1 y + \alpha_1 \) with the constants \( k_1, w_1, ..., l_1 \) satisfying

\[p(k_1, w_1, ..., l_1) = p(p_i) = 0, i = 1, 2 \text{ and } A(1, 2) = \frac{p(p_1-p_2)}{p(p_1+p_2)}.\]

**Proof:**

To construct two-soliton solution of \( F[u] = 0 \),

\[(1.11.1)\]

we take

\[f = 1 + \epsilon f_1 + \epsilon^2 f_2\]

(1.11.2)

where

\[f_1 = e^{\theta_1} + e^{\theta_2},\]  

(1.11.3)
For \( \theta_1 = k_i x + w_i t + \cdots + l_i y + \alpha_i, i = 1, 2, \)

and \( f_j = 0 \), for all \( j \geq 3 \). we shall discover what \( f_2 \) is in the process of the method. After inserting \( f \) into the equation (1.3.3), we make the coefficients of \( \epsilon^m, m = 0, 1, 2, 3, 4 \) to coefficient of \( \epsilon^0 \)

\[
p(D)[1.1] = p(0,0,...,0)\{1\} = 0
\]

gives us no information by the coefficient \( \epsilon^1 \) which is

\[
p(D)(f_1, 1 + 1. f_1) = 2p(\partial)[e^{\theta_1} + e^{\theta_2}] = 0
\]

We get \( p(p_i) = 0, i = 1, 2 \). Form the coefficient of \( \epsilon^2 \), we have

\[
p(D)(f_2, 1 + 1. f_2) + p(D)(f_1, f_1) = 2p(\partial)f_2 + p(D) \left[ (e^{\theta_1} + e^{\theta_2})(e^{\theta_1} + e^{\theta_2}) \right]
\]

\[
= 2p(\partial)f_2 + 2p(D)(e^{\theta_1} e^{\theta_2})
\]

\[
= 2p(\partial)f_2 + p(p_1 - p_2)e^{\theta_1 + \theta_2} = 0
\]

\[
2p(\partial)f_2 = -2p(p_1 - p_2)e^{\theta_1 + \theta_2}
\]

Integrating both sides for equation (1.11.6), we get:

\[
f_2 = \frac{p(p_1 - p_2)}{p(p_1 + p_2)} e^{\theta_1 + \theta_2}
\]

Since \( f_3 = 0 \) the coefficient of \( \epsilon^3 \) becomes

\[
- p(D)\{f_1.f_2 + f_2.f_1\} = A(1,2)[p(D)\{(e^{\theta_1} + e^{\theta_2}).e^{\theta_1 + \theta_2}\} + p(D)\{(e^{\theta_1} + e^{\theta_2}).e^{\theta_1 + \theta_2}\}]
\]

\[
= A(1,2)[p(D)\{(e^{\theta_1}) . e^{\theta_1 + \theta_2}\} + p(D)\{(e^{\theta_2}) . e^{\theta_1 + \theta_2}\}]
\]

\[
= A(1,2)[p(p_2)e^{2\theta_1 + \theta_2} + p(p_2)e^{\theta_1 + 2\theta_2}]
\]
which is identically zero since \( p(p_i) = 0, i = 1, 2 \). The coefficient of \( \epsilon^4 \) also vanishes trivially.

Thus \( f = 1 + e^{\theta_1} + e^{\theta_2} + A(1, 2)e^{\theta_1 + \theta_2} \)

and two-soliton solution of \( F[u] = 0 \) is

\[
u = T[f(x, t, ...)] = T[1 + e^{\theta_1} + e^{\theta_2} + A(1, 2)e^{\theta_1 + \theta_2}]
\]

Where \( \theta_i = k_i x + w_i t + \cdots + l_i y + \alpha_i \) with the constants \( k_i, w_i, ..., l_i \) satisfying

\[
p(p_i) = 0, i = 1, 2 \text{ and } A(1, 2) = -\frac{p(p_1 - p_2)}{p(p_1 + p_2)}
\]

**Theorem 2.9**

Let \( u = T[f(x, t, ..., y)] \) be a bilinearizing transformation of a nonlinear partial differential or difference equation \( F[u] = 0 \), which can be written in the Hirota bilinear form \( p(D)\{f, f\} = 0 \). Then if \( F[u] = 0 \) satisfies the three soliton condition which is

\[
\sum_{\sigma_i=\pm_1} p(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) P(\sigma_1 p_1 - \sigma_2 p_2) P(\sigma_2 p_2 - \sigma_3 p_3) P(\sigma_3 p_3 - \sigma_1 p_1) = 0
\]

With the dispersion relation \( P(p_i) = 0, i = 1, 2, 3 \)

\[
u = T[f(x, t, ...)] = T[1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1, 2)e^{\theta_1 + \theta_2} + A(1, 3)e^{\theta_1 + \theta_3} + A(2, 3)e^{\theta_2 + \theta_3} + Be^{\theta_1 + \theta_2 + \theta_3}]
\]

where

\[
\theta_i = k_i x + w_i t + \cdots + l_i y + \alpha_i, i = 1, 2, 3
\]

here

\[
A(i, j) = \frac{p(p_i - p_j)}{p(p_i + p_j)}
\]
For $i, j = 1, 2, 3, i < j$

$$B = A(1,2)A(1,3)A(2,3)$$

Proof:

To construct three-soliton solution of

$$F[u] = 0$$

We take

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3$$

where

$$f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$$

For

$$\theta_i = k_i x + w_i t + \cdots l_i y + \alpha_i, i = 1, 2, 3$$

Note that $f_j = 0$ for all $j \geq 4$. Now we insert $f$ into the equation (1.3.3) and make the coefficients of $\epsilon^m, m = 0, 1, 2, ..., 6$ to vanish. The coefficient of $\epsilon^0$ is and it is trivially zero. From the coefficient of $\epsilon^1$ which is

$$p(D)(f_1.1 + 1. f_1) = 2p(\partial)[e^{\theta_1} + e^{\theta_2} + e^{\theta_3}]$$

$$= 2[p(\partial)e^{\theta_1} + p(\partial)e^{\theta_2} + p(\partial)e^{\theta_3}] = 0$$

we have the dispersion relation $p(p_i) = 0, i = 1, 2, 3$. From the coefficient of $\epsilon^2$ we get

$$-2p(\partial)f_2 = p(D)\{f_1, f_1\}$$

where
\[ f_1 f_1 = e^{\theta_1} e^{\theta_1} + e^{\theta_2} e^{\theta_2} + e^{\theta_3} e^{\theta_3} + \sum_{i,j=1,2,3} e^{\theta_i + \theta_j} \]  

(1.11.7)

Inserting this expression into the equation (1.11.7) we obtain

\[ -2p(\partial)f_2 = 2[p(p_1 - p_2)e^{\theta_1+\theta_2} + p(p_1 - p_3)e^{\theta_1+\theta_3} + p(p_2 - p_3)e^{\theta_2+\theta_3}] \]

Hence \( f_2 \) has the form

\[ f_2 = A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} \]  

(1.11.8)

After substituting \( f_2 \) into the equation (1.11.8), we find \( A(i,j) \) as

\[ A(i,j) = \frac{p(p_i - p_j)}{p(p_i + p_j)} \]

\( i,j = 1,2,3 \ , i < j \) the coefficient of \( e^3 \) gives us

\[ -2p(\partial)f_3 = p(D)\{f_1 f_2 + f_2 f_1\} \]

\[ = 2p(D)\{f_1 f_2\} \]

where

\[ p(D)\{f_1 f_2\} = A(1,2)p(D)\{e^{\theta_1} e^{\theta_1+\theta_2} + e^{\theta_2} e^{\theta_1+\theta_2} + e^{\theta_3} e^{\theta_1+\theta_2}\} \]

\[ + A(1,3)p(D)\{e^{\theta_1} e^{\theta_1+\theta_3} + e^{\theta_2} e^{\theta_1+\theta_3} + e^{\theta_3} e^{\theta_1+\theta_3}\} \]

\[ + A(2,3)p(D)\{e^{\theta_1} e^{\theta_2+\theta_3} + e^{\theta_2} e^{\theta_2+\theta_3} + e^{\theta_3} e^{\theta_2+\theta_3}\} \]

Hence

\[ -p(\partial)f_3 = e^{\theta_1+\theta_2+\theta_3}\{A(1,2)p(p_3 - p_1 - p_2)\} \]

\[ + A(1,3)p(p_2 - p_1 - p_3) \]

\[ + A(2,3)p(p_1 - p_2 - p_3) \]

note that \( f_3 \) should have the form
\[ f_3 = B e^{\theta_1 + \theta_2 + \theta_3} \quad (1.11.9) \]

We determine \( B \) from the above equation as

\[
B = \frac{A(1,2)p(p_3 - p_1 - p_2) + A(1,3)p(p_2 - p_1 - p_3) + A(2,3)p(p_1 - p_2 - p_3)}{p(p_1 + p_2 + p_3)}
\]

Since \( f_4 = 0 \), the coefficient of \( e^4 \) becomes

\[ p(D)\{f_1\cdot f_3 + f_3\cdot f_1 + f_2\cdot f_2\} = 2p(D)\{f_1\cdot f_3\} + p(D)\{f_2\cdot f_2\} = 0 \quad (1.11.10) \]

where \( p(D)\{f_1\cdot f_3\} \) and \( p(D)\{f_2\cdot f_2\} \) are simplified as

\[
p(D)\{f_1\cdot f_3\} = B\{P(p_2 + p_3)e^{2\theta_1 + \theta_2 + \theta_3} + P(p_1 + p_3)e^{\theta_1 + 2\theta_2 + \theta_3} \\
+ P(p_1 + p_2)e^{\theta_1 + \theta_2 + 2\theta_3}\}
\]

\[
p(D)\{f_2\cdot f_2\} = 2[A(1,2)A(1,3)P(p_2 - p_3)e^{2\theta_1 + \theta_2 + \theta_3} + A(1,2)A(2,3)P(p_1 - p_3)e^{\theta_1 + 2\theta_2 + \theta_3} \\
+ A(1,3)A(2,3)P(p_1 - p_2)e^{\theta_1 + \theta_2 + 2\theta_3}]\]

Hence when we use these in equation (1.11.10) we get

\[
e^{2\theta_1 + \theta_2 + \theta_3}[BP(p_2 + p_3) + A(1,2)A(1,3)P(p_2 - p_3)] + e^{\theta_1 + 2\theta_2 + \theta_3}[BP(p_1 + p_3) + A(1,2)A(2,3)P(p_1 - p_3)] \\
+ e^{\theta_1 + \theta_2 + 2\theta_3}[B(p_1 + p_2) + A(1,3)A(2,3)P(p_1 - p_2)] = 0
\]

To satisfy the above equation, the coefficients of the exponential terms should vanish. So we find that

\[
B = A(1,2)A(1,3)A(2,3)
\]

Remember that when we are analysing the coefficient of \( e^3 \), we have found another expression for the coefficient \( B \). To be consistent these expressions for \( B \) should be equivalent i.e.

\[
B = -\frac{A(1,2)p(p_3 - p_1 - p_2) + A(1,3)p(p_2 - p_1 - p_3) + A(2,3)p(p_1 - p_2 - p_3)}{p(p_1 + p_2 + p_3)} \\
= A(1,2)A(1,3)A(2,3)
\]
when we insert the formulas for \( A(1,2), A(1,3) \) and \( A(2,3) \) in that equation, we obtain a relation that is

\[
P(p_1 - p_2)P(p_1 + p_3)P(p_1 + p_2)p(p_3 - p_1 - p_2) + P(p_1 - p_3)P(p_1 + p_2)P(p_2 + p_3)p(p_2 - p_1 - p_3) + P(p_2 - p_3)P(p_1 + p_2)P(p_1 + p_3)p(p_1 - p_2 - p_3)
\]

\[= P(p_1 - p_2)P(p_1 - p_3)P(p_2 - p_3)\]

By writing the above equation in a more appropriate form we can conclude that to have three-soliton solution, nonlinear partial differential and difference equations which have the Hirota bilinear form \( p(D)[f.f] = 0 \) should satisfy the condition which we call the three-soliton condition:

\[
\sum_{\sigma_i = -1} p(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)P(\sigma_1 p_1 - \sigma_2 p_2)P(\sigma_2 p_2 - \sigma_3 p_3)P(\sigma_3 p_3 - \sigma_1 p_1) = 0
\]

with the dispersion relation \( p(p_i) = 0, i = 1,2,3 \). an equation \( F(u) = 0 \) satisfying \((3SC)\) possesses three-soliton solution given by

\[
u = T[1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}
\]

where

\[
\theta_i = k_i x + w_i t + \cdots l_i y + \alpha_i, i = 1,2,3
\]

Here \( i, j = 1,2,3 \), \( i < j \)

and \( B = A(1,2)A(1,3)A(2,3) \). (Pekcan · 2005)
CHAPTER THREE
SOLITON SOLUTION FOR THE
KORTEWEG–DE VRIES EQUATION

3.1 Setting of the Problem:

This chapter is devoted to finding N-soliton solution of the Korteweg de Vries (KdV) equation via the Hirota direct method. We shall construct one-, two-, and three soliton solutions, after then we will generalize the idea to N-soliton solutions of the KdV.

\[ u_t - 6uu_x + u_{xxx} = 0 \]  \hspace{1cm} (2.1)

we obtain the N-soliton solution in three steps:

Step 1. we first bilinearize the equation (2.1).

\[ u(x, t) = -2 \partial^2 \log f \]

So the bilinear form of KdV is

\[ ff_{xt} - f_x f_t + ff_{xxxx} - 4f_x f_{xx} + 3f_{xx}^2 = 0 \]  \hspace{1cm} (2.2)

Step 2. Secondly we transform the above bilinear form using Hirota D-operator. Let us consider \( D_t D_x \) applied on the product \( f \cdot f \),

\[ D_t D_x \{ f \cdot f \} = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial \hat{x}} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}} \right) \{ f(x, t)f(\hat{x}, t) \}_{\hat{x} = x, t = t} \]  \hspace{1cm} (2.3)

\[ = ff_{xt} + ff_{xt} - f_x f_t - f_x f_t \]
\[ = 2( f f_{xt} - f_x f_t) \]

\[ D_\chi^4 \{ f . f \} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial \chi} \right)^4 \{ f(\tilde{x},t) f(\tilde{x},\tilde{t}) \} \quad \tilde{x}, \tilde{t} = x, t = t \]

\[ = f_{xxxx} f - 4 f_{xxx} f_x + 6 f_{xx} f_{xx} - 4 f_x f_{xxx} + f f_{xxxxx} \quad (2.4) \]

\[ = 2( f f_{xxx} - 4 f_{xxx} f_x + 3 f_{xx}^2) \]

From this and equation (2.4), we can write the equation (2.4) in the Hirota bilinear form

\[ p(D) [ f . f ] = (D_x D_t + D_\chi^4) \{ f . f \} = 0 \quad (0.5) \]

Step 3. We apply the Hirota expansion:

Insert

\[ f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \cdots \]

into (2.5), we have

\[ p(D) f . f = p(D)(1.1) + \epsilon p(D)(f_1 . 1 + 1. f_1) + \epsilon^2 p(D)(f_2 . 1 + f_1 . f_1 + 1. f_2) + \epsilon^3 p(D)(f_3 . 1 + f_2 . f_1 + f_1 . f_2 + 1. f_3) + \cdots = 0 \quad (2.6) \]

3.2 ONE-SOLUTION SOLUTION OF KDV

We take

\[ f = 1 + \epsilon f_1 \]

Where \( f_1 = e^{\theta_1} \), and \( \theta_1 = k_1 x + w_1 t + \cdots + l_1 y + \alpha_1 \), we have \( f_i = 0 \), for all \( j \geq 2 \).

We insert \( f \) into the equation (2.6), and make the coefficients of \( \epsilon^m, m = 0,1,2 \). to vanish.

The coefficient of \( \epsilon^0 \) is:

\[ p(D)[1.1] = 0 \]
Since

\[ p(D)(0,0)[1] = 0 \]

The coefficient of \( \epsilon^1 \) is:

\[
\begin{align*}
p(D)(f_1, 1 + 1. f_1) &= p(\partial)e^{\theta_1} + p(-\partial)e^{\theta_1} \\
&= 2p(\partial)e^{\theta_1} = 0
\end{align*}
\]  

(0.7)

From dispersion relation \( p(p_1) = 0 \), we get

\[
(D_xD_t + D_x^4)\{e^{\theta_1}\} = 0
\]

\[
(D_xD_t)(e^{\theta_1}) + (D_x^4)(e^{\theta_1}) = 0
\]

\[
(k_1w_1)(e^{\theta_1}) + (k_1^4)(e^{\theta_1}) = 0
\]

\[
(k_1w_1 + k_1^4)(e^{\theta_1}) = 0
\]

\[
(k_1w_1 + k_1^4) = 0
\]

\[
k_1w_1 = -k_1^4
\]

\[
w_1 = -k_1^3
\]

It is clear that the coefficient of \( \epsilon^2 \) vanishes i.e.

\[
p(D)[f_1, f_1] = p(D)[e^{\theta_1}, e^{\theta_1}] = p(p_1 - p_1)e^{2\theta_1} = 0
\]  

(0.8)

Finally, by setting \( \epsilon = 1 \), we have \( f = 1 + e^{\theta_1} \)

Then the one-soliton given by

\[
u(x, t) = -2(\log f)_{xx}
\]

\[
= -2(\log((1 + e^{\theta_1}))_{xx}
\]

\[
= -2 \frac{\partial \theta}{\partial x} \left( \frac{k_1e^{\theta_1}}{(1 + e^{\theta_1})^2} \right)
\]
\[ -2 \left( \frac{k^2 e^{\theta_1}}{(1 + e^{\theta_1})^2} \right) \] (2.9)

Figure 0-1 One Soliton Solution of KdV in three dimensions

Figure 0-2 Movement of one Soliton Solution of KdV in two dimensions
3.3 TWO-SOLITON SOLUTION OF KDV

In order to obtain the two soliton solution, we take

\[ f = 1 + \epsilon f_1 + \epsilon^2 f_2 \]

where

\[ f_1 = e^{\theta_1} + e^{\theta_2} \]

for

\[ \theta_i = k_i x + w_i t + \cdots + l_i y + \alpha_i, \quad i = 1, 2. \]

and \( f_j = 0 \), for all \( j \geq 3 \).

\( f_2 \) shall be discovered in the process of the method. We insert \( f \) into the equation (2.6) and make the coefficients of \( \epsilon^m \), \( m = 0, 1, 2, 3, 4 \)

For the coefficient of \( \epsilon^0 \)

\[ p(D)[1.1] = p(0,0,\ldots,0)\{1\} = 0 \quad (0.10) \]

For the coefficient \( \epsilon^1 \) which is

\[ (f_1, 1 + f_1) = 2p(\partial)(e^{\theta_1} + e^{\theta_2}) \]

\[ = 2[p(\partial)e^{\theta_1} + p(\partial)e^{\theta_2}] = 0 \]

which implies

\[ p(p_1) = k_i^4 + k_i w_i = 0 \Rightarrow w_i = -k_i^3, \quad \text{for} \quad i = 1, 2. \]

The coefficient of \( \epsilon^2 \) can be calculated from:

\[ p(D)(f_2, 1 + f_2 + f_1, f_1) = 2p(\partial)f_2 + p(D)[(e^{\theta_1} + e^{\theta_2})(e^{\theta_1} + e^{\theta_2})] \]

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\[ = 2[p(\partial)f_2 + p(D)(e^{\theta_1}e^{\theta_2})] \]
\[ = 2\left[p(\partial)f_2 + p(p_1 - p_2)(e^{\theta_1+\theta_2})\right] = 0 \quad (0.12) \]

This makes \( f_2 \) to have the form \( f_2 = A(1,2)e^{\theta_1+\theta_2} \)
\[ A(1,2) = \frac{p(p_1-p_2)}{p(p_1+p_2)} \frac{(k_i-k_j)^2}{(k_i+k_j)^2} \]

Since \( f_3 = 0 \), the coefficient of \( \varepsilon^3 \)
\[ p(D)(f_1,f_2,f_3) = 2A(1,2)[p(D)(e^{\theta_1})(e^{\theta_1+\theta_2}) + p(D)(e^{\theta_2})(e^{\theta_1+\theta_2})] \]
\[ = 2 \left[ A(1,2)p(p_2)(e^{2\theta_1+\theta_2}) + (p_1)(e^{\theta_2})(e^{\theta_1+2\theta_2}) \right] \quad (0.13) \]

and this is already zero since
\[ p(p_i) = 0, i = 1,2. \]

now let us set \( \varepsilon = 1 \).

thus
\[ f = 1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2} \]

and two-soliton solution of kdv is
\[ u(x,t) = -2(\log f)_{xx} \]
\[ = -2(\log(1 + e^{\theta_1} + e^{\theta_2} + A(1,2)e^{\theta_1+\theta_2}))_{xx} \quad (3.14) \]
\[ u(x,t) = \frac{-2[K_1e^{\theta_1}+K_2e^{\theta_2}+A(1,2)(K_1^2+K_2^2)e^{\theta_1+\theta_2}+2(k_1-k_2)^2e^{\theta_1+\theta_2}]}{(1+e^{\theta_1}+e^{\theta_2}+A(1,2)e^{\theta_1+\theta_2})^2} \quad (0.15) \]
Figure 0-3 two Soliton Solution in three dimensions
Figure 0-4: Two Soliton Solution in two dimensions

Figure 0-5: Two Soliton Solution in two dimensions

Figure 0-6: Two Soliton Solution in two dimensions
3.4 THREE-SOLITON SOLUTION OF KDV

We take

\[ f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 \]

where

\[ f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \]

and \( \theta_1 = k_1 x + w_1 t + \cdots + l_i y + \alpha_i, i = 1, 2, 3. \)

Note that \( f_j = 0 \) for all \( j \geq 4. \) Now we insert \( f \) into the equation (2.6) and make the coefficients of \( \epsilon^m, m = 0, 1, 2, \ldots, 6 \) to vanish. The coefficient of \( \epsilon^0 \) is zero.

For \( \epsilon^1 \), we have

\[ p(D)(f_1, 1 + f_1) = 2p(\partial)(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) = 0. \]  \hspace{1cm} \text{(0.16)}

For \( \epsilon^2 \), we get

\[ p(D)(f_2, 1 + f_2 + f_1 f_1) = 0 \]

\[ -p(\partial)f_2 = (D_x D_t + D_x^4)(f_1, f_1) \]

\[ -p(\partial)f_2 = (D_x D_t + D_x^4)(e^{\theta_1} + e^{\theta_2} + e^{\theta_3})(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) \]

\[ = [(k_1 - k_2)(w_1 - w_2) + (k_1 - k_2)^4]e^{\theta_1} + [k_1 - k_3)(w_1 - w_3) + (k_1 - k_3)^4]e^{\theta_1 + \theta_2} + [(k_1 - k_3)(w_1 - w_3) + (k_1 - k_3)^4]e^{\theta_1 + \theta_2 + \theta_3} \]  \hspace{1cm} \text{(0.17)}

Integrating both sides of equation (2.17), we get:

\[ f_2 = \]

\[ \frac{[(k_1 - k_2)(w_1 - w_2) + (k_1 - k_2)^4]e^{\theta_1 + \theta_2}}{[(k_1 + k_2)(w_1 + w_2) + (k_1 + k_2)^4] + [k_1 - k_3)(w_1 - w_3) + (k_1 - k_3)^4]} + \]

\[ \frac{[(k_1 - k_3)(w_1 - w_3) + (k_1 - k_3)^4]e^{\theta_1 + \theta_2 + \theta_3}}{[(k_1 + k_3)(w_1 + w_3) + (k_1 + k_3)^4]} \]  \hspace{1cm} \text{(2.18)}

\[ f_2 = A(1,2)e^{\theta_1 + \theta_2} + A(1,3)e^{\theta_1 + \theta_3} + A(2,3)e^{\theta_2 + \theta_3} \]
where  \[ A(i,j) = \frac{p(p_i-p_j)}{p(p_i+p_j)} = \frac{p(k_i-k_j)^2}{p(k_i+k_j)^2} \]  \( (0.19) \)

For \( i,j = 1,2,3, i < j \).

For \( \epsilon^3 \), we get

\[ p(D)(f_3.1 + f_2.f_1 + f_1.f_2 + 1.f_3) = 0 \]

\[ -2p(\partial)f_3 = (D_xD_t + D_x^4)(f_2.f_1 + f_1.f_2) \]

\[ = 2(D_xD_t + D_x^4)[(A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3})[e^{\theta_1} + e^{\theta_2} + e^{\theta_3}] \]

\[ = e^{\theta_1+\theta_2+\theta_3} \{ A(1,2)p(p_3 - p_2 - p_1) + A(1,3)p(p_2 - p_1 - p_3) + A(2,3)p(p_1 - p_2 - p_3) \} \]

(0.20)

Integrating both sides of equation (2.20), we get:

\[ f_3 = Be^{\theta_1+\theta_2+\theta_3} \]

where

\[ B = \frac{A(1,2)p(p_3 - p_1 - p_2) + A(1,3)p(p_2 - p_1 - p_3) + A(2,3)p(p_1 - p_2 - p_3)}{p(p_1 + p_2 + p_3)} \]

For \( \epsilon^4 \)

\[ p(D)(f_4.1 + f_3.f_1 + f_2.f_2 + f_1.f_3 + 1.f_4) = 0 \]

since \( f_4 = 0 \) we get

\[ p(D)(f_3.f_1 + f_2.f_2 + f_1.f_3) = 0 \]

(0.21)

\[ p(D)((Be^{\theta_1+\theta_2+\theta_3})(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) + (A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3})(A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}) + (Be^{\theta_1+\theta_2+\theta_3}) = 0 \]
\[ e^{2\theta_1 + \theta_2 + \theta_3} [B(p_2 + p_3) + A(1,2)A(1,3)(p_2 - p_3) + e^{\theta_1 + 2\theta_2 + \theta_3} [B(p_1 + p_3) + A(1,2)A(2,3)(p_1 - p_3)] + e^{\theta_1 + \theta_2 + 2\theta_3} [B(p_1 + p_2) + A(1,3)A(2,3)B(p_1 - p_2) = 0 \]

Finally for \( \epsilon^5 \) and \( \epsilon^6 \) is vanish.

We may also \( \epsilon = 1 \), therefore

\[
 f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2) e^{\theta_1 + \theta_2} + A(1,3) e^{\theta_1 + \theta_3} + A(1,3) e^{\theta_2 + \theta_3} + B e^{\theta_1 + \theta_2 + \theta_3}
\]

So three-soliton solution of KdV is

\[
 u(x, t) = -2 \left( \log f \right)_{xx}
\]

\[
 = -2 \left( \log(1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2) e^{\theta_1 + \theta_2} + A(1,3) e^{\theta_1 + \theta_3} + A(1,3) e^{\theta_2 + \theta_3} + B e^{\theta_1 + \theta_2 + \theta_3}) \right)_{xx}
\]

\[
 u(x, t) = -2 \frac{R(x, t)}{N(x, t)}
\]

where

\[
 R(x, t) = e^{\theta_1 + \theta_2} [2(k_1 - k_2)^2 + 2(k_1 - k_2)^2 A(1,3)A(2,3) e^{2\theta_3} + A(1,2)k_1^2 e^{\theta_2} + A(1,2)k_2^2 e^{\theta_1}] + e^{\theta_1 + \theta_2} [2(k_1 - k_3)^2 + 2(k_1 - k_3)^2 A(1,3)A(2,3) e^{2\theta_2} + A(1,2)k_1^2 e^{\theta_3} + A(1,2)k_3^2 e^{\theta_1} + e^{\theta_2 + \theta_3} [2(k_2 - k_3)^2 + 2(k_2 - k_3)^2 A(1,3)A(2,3) e^{2\theta_1} + A(1,2)k_2^2 e^{\theta_3} + A(1,3)k_2^2 e^{\theta_1 + \theta_2} + A(2,3)k_1^2 e^{\theta_2 + \theta_3} + e^{\theta_1 + \theta_2 + \theta_3} [A(1,2)(k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 - 2k_1k_3 - 2k_2k_3) + A(1,3)(k_2^2 + k_2^2 + k_3^2 + 2k_2k_3 - 2k_1k_2 - 2k_1k_3) + B(k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 + 2k_1k_3 + 2k_2k_3)].
\]
and

\[ N(x, t) = [1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} + B e^{\theta_1+\theta_2+\theta_3}]^2 \]

for \( \theta_i = k_i x - k_i^2 t + \alpha_i \)

\[ A(i, j) = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, i, j = 1, 2, 3, i < j, \]

\[ B = A(1,2) A(1,3) A(2,3). \]

Figure 0-3 three_Solution in three dimensions
Fig. 2.10 Solution in two dimensions

Fig. 2.11 Solution in two dimensions

Fig. 2.12 Solution in two dimensions
3.5 N-SOLITON SOLUTION OF KDV:

The bilinear form of KdV is

\[ ff_{xt} - f_x f_t + f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx}^2 = 0 \]

For N-soliton solution of KdV, we claim that \( f(x,t) \) takes the form

\[ f(x,t) = 1 + \sum_{m=1}^{N} \sum_{N=m}^{N} A(i_1, ..., i_m) \exp(\theta_{i_1}, ..., \theta_{i_m}) \]

where \( A(i_1, ..., i_m) = \prod_{l<j}^{(m)} A(l,j) \), \( A(l,j) = \frac{(k_l - k_j)^2}{(k_l + k_j)^2} \). 

Here \( NC_m \) indicates the summation over all possible combinations of \( m \) elements from \( N \) and \( (m) \) indicates the product of all possible combinations of the \( m \) elements with \( l < j \). Note that \( A(i_m) = 1 \) for \( m = 1, 2, ..., N \). To prove our claim we substitute the expression for \( f(x,t) \) into (3.26) and see whether it is satisfied. Substitution of \( f(x,t) \) gives us some exponential terms. To satisfy the bilinear form of KdV the coefficients of the exponential terms should vanish. From these coefficients we get the relation

\[ \sum_{r=0}^{m} \sum_{mC_r} A(i_1, ..., i_r) A(i_{r+1}, ..., i_m) g(i_1, ..., i_r; i_{r+1}, ..., i_m) , m = 1, 2, ..., N \]

where

\[ g(i_1, ..., i_r; i_{r+1}, ..., i_m) = (-k_{i_1} - ... - k_{i_r} + k_{i_{r+1}} + ... + k_{i_m}) \times [(-k_{i_1} - ... - k_{i_r} + k_{i_{r+1}} + ... + k_{i_m})^3 - (-k_{i_1}^3 - ... - k_{i_r}^3 + k_{i_{r+1}}^3 + ... + k_{i_m}^3)] \] (Pekcan, 2005)
CHAPTER FOUR

SOLUTION FOR BOUSSINESQ EQUATION

In this chapter, we investigate the behaviour of the solitary waves on deep waters; this is described by the Boussinesq equation:

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0,$$ (3.1)

4.1 DERIVING AN EXPONENTIAL FORM FOR THE SOLITON OF BOUSSINESQ EQUATION

To achieve what we aim in this section we use:

$$u = 2 \log f_{xx}$$

Applying the assumed solution into (3.1), we get

$$2(\log f)_{tt} - 2(\log f)_{xx} - 3[2(\log f_{xx})^2 - 2(\log f)_{xx}] = 0$$

$$\left(\frac{D_t^2 f.f}{f^2}\right) - \left(\frac{D_x^2 f.f}{f^2}\right) - 3\left(\frac{D_x^2 f.f}{f^2}\right)^2 - \left(\frac{D_t^2 f.f}{f^2}\right) + 3\left(\frac{D_x^4 f.f}{f^2}\right)^2 = 0$$ (3.2)

Multiplying both sides of equations (3.2) by $f^2$, we get

$$D_t^2 (f.f) - D_x^2 (f.f) - D_x^4 (f.f) = 0$$ (3.3)

$$p(D)(f.f) = (D_t^2 - D_x^2 - D_t^4)(f.f) = 0$$ (3.4)

Now application of the Hirota perturbation:

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots$$

Into the equation (3.4) we have
\[
(D_t^2 - D_x^2 - D_t^4)(1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots)(1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots) = 0
\]

\[
p(D)(1.1) + \epsilon p(D)(f_1.1 + 1. f_1) + \epsilon^2 p(D)(f_2.1 + f_1. f_1 + 1. f_2) + \epsilon^3 p(D)(f_3.1 + f_2. f_1 + f_1. f_2 + 1. f_3) + \cdots = 0
\]  

(3.5)

### 4.2 ONE-SOLITON SOLUTION OF BOUSSINESQ EQUATION:

We take \( f = 1 + \epsilon f_1 \) where \( f_1 = e^{\theta_1} \), and \( \theta_1 = k_1 x + w_1 t \), we have \( f_i = 0 \), for all \( j \geq 2 \). We insert \( f \) into the equation (3.4), and make the coefficients of \( \epsilon^m, m = 0, 1, 2 \) to vanish.

For \( \epsilon^0 \), we have

\[
p(D)[1.1] = 0
\]

Since

\[
p(D)(0,0)[1] = 0
\]

For \( \epsilon^1 \), we have

\[
p(D)(f_1, 1 + 1. f_1) = p(\partial)e^{\theta_1} + p(-\partial)e^{\theta_1} = 2p(\partial)e^{\theta_1} = 0
\]

have the dispersion relation \( p(p_1) = 0 \), which implies

\[
(D_t^2 - D_x^2 - D_t^4)(e^{\theta_1}) = 0
\]

\[
(w_1^2 - k_1^2 - k_1^4)(e^{\theta_1}) = 0
\]

\[
w_1^2 = k_1^2 + k_1^4
\]

\( f = 1 + e^{\theta_1} \) then one-soliton solution

\[
u = 2 \frac{\partial}{\partial x}(\log f_{xx} = 2(\log((1 + e^{\theta_1}))_{xx})
\]

\[
= 2 \frac{\partial}{\partial x} \left( k_1 e^{\theta_1} \right)
\]

\[
\frac{1}{(1 + e^{\theta_1})^2}
\]

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\[ = 2 \left( \frac{k^2_1 e^{\theta_1}}{(1 + e^{\theta_1})^2} \right) \]

*Figure 4-1 One Soliton Solution of Boussinesq equation in three dimensions*
Figure 4-2 One Soliton Solution of Boussinesq equation in two dimensions

Figure 4-3 One Soliton Solution of Boussinesq equation in two dimensions

Figure 4-4 One Soliton Solution of Boussinesq equation in three dimensions
4.3 **Two-Soliton Solution of Boussinesq Equation:**

In order to obtain the two soliton solution, we take

\[ f = 1 + \epsilon f_1 + \epsilon^2 f_2 \]

where

\[ f_1 = e^{\theta_1} + e^{\theta_2} \]

For

\[ \theta_1 = k_l x + w_i t + \cdots + l_i y + \alpha_i, i = 1, 2, \]

and \( f_j = 0 \), for all \( j \geq 3 \).

\( f_2 \) shall be discovered in the process of the method. We insert \( f \) into the equation (3.1.4) and make the coefficients of \( \epsilon^m \), \( m = 0, 1, 2, 3, 4 \)

For the coefficient of \( \epsilon^0 \)

\[ p(D)[1.1] = p(0,0,...,0)(1) = 0 \]  \hspace{1cm} (3.6)

Gives us no information by the coefficient \( \epsilon^1 \) which is

\[ (f_1, 1 + 1, f_1) = 2p(\partial)(e^{\theta_1} + e^{\theta_2}) \]

\[ = 2[p(\partial)e^{\theta_1} + p(\partial)e^{\theta_2}] = 0 \]  \hspace{1cm} (3.7)

The coefficient of \( \epsilon^2 \)

\[ p(D)(f_2, 1 + 1, f_2 + f_1, f_1) = 2p(\partial)f_2 + p(D)[(e^{\theta_1} + e^{\theta_2})(e^{\theta_1} + e^{\theta_2})] \]

\[ = 2[p(\partial)f_2 + (D^2 - D^2)(e^{\theta_1}e^{\theta_2})] \]

\[ = 2[p(\partial)f_2 + (p_1 - p_2)(e^{\theta_1 + \theta_2})] = 0 \]

This makes \( f_2 \) to have the form \( f_2 = A(1,2)e^{\theta_1 + \theta_2} \)
\[ B(1,2) = \frac{p(p_1-p_2)}{p(p_1+p_2)} \] (3.8)

Since \( f_3 = 0 \), the coefficient of \( \epsilon^3 \)

\[
p(D)(f_1.f_2+ f_2.f_3) = 2 B(1,2)[p(D)(e^{\theta_1}) (e^{\theta_1+\theta_2}) + p(D)(e^{\theta_2}) (e^{\theta_1+\theta_2})] \\
= 2 [B(1,2)p(p_2)(e^{2\theta_1+\theta_2}) + p(p_1)(e^{\theta_2})(e^{\theta_1+2\theta_2})] \] (3.9)

and this is already zero since \( p(p_i) = 0, i = 1,2. \)

For \( \epsilon^4 \) also vanishes trivially. At last we may set \( \epsilon = 1 \),

thus \( f = 1 + e^{\theta_1} + e^{\theta_2} + B(1,2)e^{\theta_1+\theta_2} \),

and tow-soliton of Boussinesq equation:

\[
\begin{align*}
\frac{u(x,t)}{u(x,t)} &= 2(\log f)_{xx} \\
&= 2(\log(1 + e^{\theta_1} + e^{\theta_2} + B(1,2)e^{\theta_1+\theta_2}))_{xx} \quad (3.10)
\end{align*}
\]

\[
\begin{align*}
\frac{u(x,t)}{u(x,t)} &= \frac{-2[k_1e^{\theta_1+k_2e^{\theta_2}}+A(1,2)(k_1e^{\theta_1+k_2e^{\theta_2}})e^{\theta_1+\theta_2}+2(k_1-k_2)^2e^{\theta_1+\theta_2}]}{(1+e^{\theta_1+\theta_2}+A(1,2)e^{\theta_1+\theta_2})^2} \quad (3.11)
\end{align*}
\]

\textit{Figure 4-5 Two Soliton Solution of Boussinesq equation in three dimensions}
Figure 4-6 Two Soliton Solution of Boussinesq equation in three dimensions

Figure 4-7 Two Soliton Solution of Boussinesq equation in two dimensions

Figure 4-8 Two Soliton Solution of Boussinesq equation in two dimensions
4.4 THREE- SOLITON SOLUTION OF BOUSSINESQ EQUATION:

We take,

\[ f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 \]

where \( f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \) and \( \theta_i = k_i x + w_i t, i = 1,2,3 \).

note that \( f_j = 0 \) for all \( j \geq 4 \). Now we insert \( f \) into the equation (3.5) and

Make the coefficients of \( \epsilon^m, m = 0,1,2,\ldots 6 \) to vanish. The coefficient of \( \epsilon^0 \) is zero.

For \( \epsilon^1 \), we have

\[ p(D)(f_1 \cdot 1 + 1. f_1) = 2p(\partial)(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) = 0 \quad (3.12) \]

For \( \epsilon^2 \), we get

\[ p(D)(f_2 \cdot 1 + 1. f_2 + f_1 \cdot f_1) = \]

\[ -p(\partial)f_2 = (D_x^2 - D_x^2 - D_x^4)(f_1 \cdot f_1) \]

\[ -p(\partial)f_2 = (D_x^2 - D_x^2 - D_x^4)(e^{\theta_1} + e^{\theta_2} + e^{\theta_3})(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) \]

\[ = [(w_1 - w_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4]e^{\theta_1 + \theta_2} + [(w_1 - w_3)^2 - (k_1 - k_3)^2 - (k_1 - k_3)^4]e^{\theta_1 + \theta_3} + [(w_2 - w_3)^2 - (k_2 - k_3)^2 - (k_2 - k_3)^4]e^{\theta_2 + \theta_3} \quad (3.13) \]

Integrating both sides for equation (3.13), we get:

\[ f_2 = \frac{[(w_1 - w_2)^2 - (k_1 - k_2)^2 - (k_1 - k_2)^4]e^{\theta_1 + \theta_2}}{[(w_1 + w_2)^2 - (k_1 + k_2)^2 - (k_1 + k_2)^4]} + \frac{[(w_1 - w_3)^2 - (k_1 - k_3)^2 - (k_1 - k_3)^4]e^{\theta_1 + \theta_3}}{[(w_1 + w_3)^2 - (k_1 + k_3)^2 - (k_1 + k_3)^4]} + \]

\[ + \frac{[(w_2 - w_3)^2 - (k_2 - k_3)^2 - (k_2 - k_3)^4]e^{\theta_2 + \theta_3}}{[(w_2 + w_3)^2 - (k_2 + k_3)^2 - (k_2 + k_3)^4]} \quad (3.14) \]

\[ f_2 = B(1,2)e^{\theta_1 + \theta_2} + B(1,3)e^{\theta_1 + \theta_3} + B(2,3)e^{\theta_2 + \theta_3} \]

Where

\[ B(i,j) = \frac{p(p_i-p_j)}{p(p_i+p_j)} \quad (3.15) \]

For \( i,j = 1,2,3, i < j \).
For $\epsilon^3$, we get

\[ p(D)(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0 \]

\[-2p(\partial)f_3 = (D^2_t - D^2_x - D^4_\xi)(f_2 \cdot f_1 + f_1 \cdot f_2) \]

\[ = 2(D^2_t - D^2_x - D^4_\xi)\{[B(1,2)e^{\theta_1 + \theta_2} + B(1,3)e^{\theta_1 + \theta_3} + B(2,3)e^{\theta_2 + \theta_3}][e^{\theta_1} + e^{\theta_2} + e^{\theta_3}] \}
\]

\[ = e^{\theta_1 + \theta_2 + \theta_3}\{ B(1,2)p(p_3 - p_2 - p_1) + B(1,3)p(p_2 - p_1 - p_3) + B(2,3)p(p_1 - p_2 - p_3) \} \quad (3.16) \]

Integrating both sides for equation (3.16), we get:

\[ f_3 = Ae^{\theta_1 + \theta_2 + \theta_3} \]

where

\[ A = \frac{B(1,2)p(p_3 - p_1 - p_2) + B(1,3)p(p_2 - p_1 - p_3) + B(2,3)p(p_1 - p_2 - p_3)}{p(p_1 + p_2 + p_3)} \]

For $\epsilon^4$

\[ p(D)(f_4 \cdot 1 + f_3 \cdot f_1 + f_2 \cdot f_2 + f_1 \cdot f_3 + 1 \cdot f_4) = 0 \]

since $f_4 = 0$ we get

\[ p(D)(f_3 \cdot f_1 + f_2 \cdot f_2 + f_1 \cdot f_3) = 0 \quad (3.17) \]

\[ p(D)\{(Be^{\theta_1 + \theta_2 + \theta_3})(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) \]

\[ + (B(1,2)e^{\theta_1 + \theta_2} + B(1,3)e^{\theta_1 + \theta_3} + B(2,3)e^{\theta_2 + \theta_3})(B(1,2)e^{\theta_1 + \theta_2} + e^{\theta_1 + \theta_3} + B(2,3)e^{\theta_2 + \theta_3}) \]

\[ = 0 \]

\[ e^{2\theta_1 + \theta_2 + \theta_3}[A(p_2 + p_3) + B(1,2)A(1,3)] \]
\[(p_2 - p_3) + e^{\theta_1 + 2\theta_2 + \theta_3}[B(p_1 + p_3) + B(1,2)A(2,3)(p_1 - p_2)]
+ e^{\theta_1 + \theta_2 + 2\theta_3}[A(p_1 + p_2) + B(1,3)A(2,3)A(p_1 - p_2) = 0
\]

Finally for \(\epsilon^5\) and \(\epsilon^6\) is vanish, we may also \(\epsilon = 1\), therefore
\[
f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + B(1,2)e^{\theta_1 + \theta_2} + B(1,3)e^{\theta_1 + \theta_3} + B(1,3)e^{\theta_2 + \theta_3} + Ae^{\theta_1 + \theta_2 + \theta_3}
\]

So Three- soliton solution of Boussinesq equation is:
\[
u(x, t) = 2(log f)_{xx}
= 2\left(\log(1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + B(1,2)e^{\theta_1 + \theta_2} + B(1,3)e^{\theta_1 + \theta_3} + B(1,3)e^{\theta_2 + \theta_3} + Ae^{\theta_1 + \theta_2 + \theta_3})\right)_{xx}
\]

Figure 4-9 Three_Soliton Solution of Boussinesq equation in three dimensions
Figure 4-10 Three Soliton Solution of Boussinesq equation in two dimensions

Figure 4-11 Three Soliton Solution of Boussinesq equation in two dimensions

Figure 4-12 Three Soliton Solution of Boussinesq equation in two dimensions
CHAPTER FIVE
SHALLOW WATER WAVES EQUATION

In this chapter, we study the behavior of the solitary waves in shallow waters. This is described by the shallow water wave equation:

\[ u_t + u_{xxt} - 3uu_x - 3u_x \int u_t \, dx + u_x = 0, \quad (4.1) \]

Where we use

\[ u(x, t) = 2(\log f(x, t))_{xx} \]

5.1 DERIVING ONE-SOLITON SOLUTION TO THE SHALLOW WATER WAVES EQUATION:

Substituting \( u(x, t) = w_x(x, t) \) into equation (4.1), we get:

\[ w_{xt} + w_{xxt} - 3w_xw_{xt} - 3w_{xx}w_t + w_x = 0 \quad (4.2) \]

Substituting

\[ w(x, t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t \]

such that

\[ w_x = k_i e^{\theta_i} \rightarrow w_{xx} = k_i^2 e^{\theta_i} \rightarrow w_{xxx} = k_i^3 e^{\theta_i} \rightarrow w_{xxt} = k_i^3 c_i e^{\theta_i} \]

\[ w_{xt} = -k_i c_i e^{\theta_i} \]

we obtain

\[ -k_i c_i e^{\theta_i} + k_i^3 c_i e^{\theta_i} + k_i^2 e^{\theta_i} = 0 \]

then

\[ c_i = \frac{k_i}{1-k_i^2} \quad (4.3) \]

We obtain One-soliton Shallow water waves equation:

\[ u(x, t) = w_x(x, t) = 2(\log f)_{xx} \rightarrow w(x, t) = 2(\log f)_x = \frac{2f_x}{f} \]

where

\[ f = 1 + e^{\frac{k_i x - k_i t}{1-k_i^2}} \]
\[ f_x = k_1 e^{\frac{k_1 x - k_1 t}{1-k_1^2}} \]

\[ w(x,t) = \frac{2k_1 e^{\frac{k_1 x - k_1 t}{1-k_1^2}}}{1 + e^{\frac{k_1 x - k_1 t}{1-k_1^2}}} \]

\[ u(x,t) = w_x(x,t) = \frac{2k_1^2 e^{\frac{k_1 x - k_1 t}{1-k_1^2}}}{\left(1 + e^{\frac{k_1 x - k_1 t}{1-k_1^2}}\right)^2} \] (4.4)

Figure 5-1 Three Soliton Solution of Shallow water waves equation in three dimensions
Figure 5-2 One Soliton Solution of Shallow water waves equation in two dimensions

Figure 5-3 One Soliton Solution of Shallow water waves equation in two dimensions

Figure 5-4 One Soliton Solution of Shallow water waves equation in two dimensions
For two solutions we get

\[ f_1 = e^{\theta_1} + e^{\theta_2} \]
\[ f_2 = e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \]
\[ a_{12} = \frac{(k_1^2 - k_1 k_2 - k_2^2 - 3)(k_1 - k_2)^2}{(k_1^2 - k_1 k_2 + k_2^2 - 3)(k_1 + k_2)^2} \]  
(4.5)

\[ a_{ij} = \frac{(k_i^2 - k_i k_j - k_j^2 - 3)(k_i - k_j)^2}{(k_i^2 - k_i k_j + k_j^2 - 3)(k_i + k_j)^2} \]

\[ f(x, t) = 1 + e^{k_1 x - \frac{k_1 k_2}{1 - k_1^2} t} + e^{k_2 x - \frac{k_2 k_1}{1 - k_2^2} t} + \left( \frac{k_1^2 - k_1 k_2 - k_2^2 - 3}{(k_1 - k_1 k_2 + k_2^2 - 3)(k_1 + k_2)^2} \right) e^{k_1 x - \frac{k_1}{1 - k_1^2} t + k_2 x - \frac{k_2}{1 - k_2^2} t} \]  
(4.6)

5.2 TWO-SOLITON SOLUTION OF SHALLOW WATER WAVES EQUATION:

To determine the two solutions we substitute (4.6) into the formula

\[ u(x, t) = 2(\log f)_{xx} \]

We get

\[ u(x, t) = 2(\log f)_{xx} \]

\[ \frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial^2 f}{\partial x^2} \]

Figure 5-5 Two Soliton Solution of Shallow water waves equation in three dimensions
Figure 5-6 Two Soliton Solution of Shallow water waves equation in two dimensions

Figure 5-7 Two Soliton Solution of Shallow water waves equation in two dimensions

Figure 5-8 Two Soliton Solution of Shallow water waves equation in two dimensions
For the three-soliton solutions, we set
\[ f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + A_{123}e^{\theta_1+\theta_2+\theta_3} \]
\[ A_{123} = a_{12}a_{13}a_{23} \]  

5.3 **THREE-SOLITON SOLUTION OF SHALLOW WATER WAVES EQUATION:**

To determine the three-soliton solutions, we substitute (4.1.26) into the Formula

\[ u(x, t) = 2(\log f)_{xx} \]

![Figure 5-9 Three_Soliton Solution of Shallow water waves equationin in three dimensions](image)
Figure 5-10 Three Soliton Solution of Shallow water waves equation in two dimensions

Figure 5-11 Three Soliton Solution of Shallow water waves equation in two dimensions

Figure 5-12 Three Soliton Solution of Shallow water waves equation in two dimensions
6.1 **Conclusion:**

We have shown that the waves on shallow waters have two main properties these are: small wavelength and slower speed, in opposite to the waves on deep waters are always faster with large wavelength. We have used mathematic programming to show how solitary waves with long height come across shorter waves and pass them without any effect. Up to our knowledge this investigations and disctions are new.

6.2 **Recommendation:**

We recommend that solitary waves in nonlinear PDEs with more complicated nonlinearity to be investigated.
REFERENCES


Anon., n.d. s.l.:s.n.

Druitt, F., 2005. Hirota’s direct method and Sato’s formalism in soliton theory.

Hietarinta, Jarmo, 1987. *A search for bilinear equations passing Hirota’s three-soliton condition. II. mKdV-type bilinear equations*. s.l.:AIP.


One-solution solution of kdv:

\[ p = 0.7; \]
\[ f[x_,t_] = 1 + \exp[(p) x - (p^3) t]; \]
\[ u[x_,t_] = 2D[\log[f[x,t]],x,x]; \]
\[ \text{Plot3D}[u[x,t], \{x,-20,20\}, \{t,-20,20\}, \text{PlotRange} \to \{-0.01,1\}] \]

\[ u[x_] = 2D[\log[f[x,10]],x,x]; \]
\[ \text{Plot}[u[x], \{x,-60,60\}, \text{Filling} \to \text{Bottom}, \text{PlotRange} \to \{-60,60\}, \{0,0.6\}] \]

\[ u[x_] = 2D[\log[f[x,30]],x,x]; \]
\[ \text{Plot}[u[x], \{x,-60,60\}, \text{Filling} \to \text{Bottom}, \text{PlotRange} \to \{-60,60\}, \{0,0.6\}] \]

\[ u[x_] = 2D[\log[f[x,-50]],x,x]; \]
\[ \text{Plot}[u[x], \{x,-60,60\}, \text{Filling} \to \text{Bottom}, \text{PlotRange} \to \{-60,60\}, \{0,0.6\}] \]

// //

Two-solution solution of kdv:

\[ p1 = 1; \]
\[ p2 = 0.7; \]
\[ A12 = (p1-p2)^2/(p1+p2)^2; \]
\[ f[x_,t_] = 1 + \exp[(p1) x - (p1^3) t] + \exp[(p2) x - (p2^3) t] + A12 \exp[(p1+p2) x - (p1^3+p2^3) t]; \]
\[ u[x_,t_] = -2D[\log[f[x,t]],x,x]; \]
\[ \text{Plot3D}[u[x,t], \{x,-15,15\}, \{t,-15,15\}, \text{BoxRatios} \to \{1,1,0.1\}] \]

\[ u[x_] = 2D[\log[f[x,10]],x,x]; \]
\[ \text{Plot}[u[x], \{x,-60,60\}, \text{Filling} \to \text{Bottom}, \text{PlotRange} \to \{-60,60\}, \{0,0.6\}] \]

\[ u[x_] = 2D[\log[f[x,30]],x,x]; \]
\[ \text{Plot}[u[x], \{x,-60,60\}, \text{Filling} \to \text{Bottom}, \text{PlotRange} \to \{-60,60\}, \{0,0.6\}] \]

\[ u[x_] = 2D[\log[f[x,-50]],x,x]; \]
\[ \text{Plot}[u[x], \{x,-60,60\}, \text{Filling} \to \text{Bottom}, \text{PlotRange} \to \{-60,60\}, \{0,0.6\}] \]
Three-soliton solution of KDV:

\[
\begin{align*}
\text{p1} &= 1; \\
\text{p2} &= 0.8; \\
\text{p3} &= 0.6; \\
\text{A12} &= \frac{(p1-p2)^2}{(p1+p2)^2}; \\
\text{A13} &= \frac{(p1-p3)^2}{(p1+p3)^2}; \\
\text{A23} &= \frac{(p2-p3)^2}{(p2+p3)^2}; \\
\text{f}[x_,t_] &= 1 + \exp[(p1)x-(p1^3)t] + \exp[(p2)x-(p2^3)t] + \exp[(p3)x-(p3^3)t] + \text{A12}\exp[(p1)x-(p1^3)t+(p2)x-(p2^3)t] + \text{A13}\exp[(p1)x-(p1^3)t+(p3)x-(p3^3)t] + \text{A23}\exp[(p2)x-(p2^3)t+(p3)x-(p3^3)t] + \text{A12}\text{A13}\exp[(p1+p2)x-(p1^3+p2^3)t]; \\
\text{u}[x_,t_] &= -2D[\text{Log}[f[x,t]],x,x]; \\
\text{Plot3D}[u[x,t],[x,-15,15],[t,-15,15],\text{BoxRatios}\rightarrow\{1,1,0.2\}]
\end{align*}
\]

One-soliton solution of Boussinesq equation:

\[
\begin{align*}
\text{p1} &= 0.7; \\
\text{f}[x_,t_] &= 1 + \exp[(p1)x-Sqrt[p1^2+p1^4]*t]; \\
\text{u}[x_,t_] &= 2D[\text{Log}[f[x,t]],x,x]; \\
\text{Plot3D}[u[x,t],[x,-10,10],[t,-10,10]]
\end{align*}
\]
Two-soliton solution of Boussinesq equation:

\[ p_1 = 1; \]
\[ p_2 = 0.7; \]
\[ \Delta_{12} = (p_1 - p_2)^2 / (p_1 + p_2)^2; \]
\[ f[x_, t_] = 1 + \exp[(p_1) x - \sqrt{p_1^2 + p_1^4} t] + \exp[(p_2) x - \sqrt{p_2^2 + p_2^4} t] + \Delta_{12} \exp[(p_1) x - \sqrt{p_1^2 + p_1^4} t + (p_2) x - \sqrt{p_2^2 + p_2^4} t]; \]
\[ u[x_, t_] = \log[f[x, t]]; \]
\[ \text{Plot3D}[u[x, t], \{x, -15, 15\}, \{t, -15, 15\}] \]

Three-soliton solution of Boussinesq equation:

\[ p_1 = 1; \]
\[ p_2 = 0.8; \]
\[ p_3 = 0.6; \]
\[ \Delta_{12} = (p_1 - p_2)^2 / (p_1 + p_2)^2; \]
\[ \Delta_{13} = (p_1 - p_3)^2 / (p_1 + p_3)^2; \]
\[ \Delta_{23} = (p_2 - p_3)^2 / (p_2 + p_3)^2; \]
\[ f[x_, t_] = 1 + \exp[(p_1) x - \sqrt{p_1^2 + p_1^4} t] + \exp[(p_2) x - \sqrt{p_2^2 + p_2^4} t] + \exp[(p_3) x - \sqrt{p_3^2 + p_3^4} t] + \Delta_{12} \exp[(p_1) x - \sqrt{p_1^2 + p_1^4} t + (p_2) x - \sqrt{p_2^2 + p_2^4} t] + \Delta_{13} \exp[(p_1) x - \sqrt{p_1^2 + p_1^4} t + (p_3) x - \sqrt{p_3^2 + p_3^4} t] + \Delta_{23} \exp[(p_2) x - \sqrt{p_2^2 + p_2^4} t + (p_3) x - \sqrt{p_3^2 + p_3^4} t] + \Delta_{12} \Delta_{13} \exp[(p_1 + p_2 + p_3) x - (\sqrt{p_1^2 + p_1^4} + \sqrt{p_2^2 + p_2^4} + \sqrt{p_3^2 + p_3^4}) t]; \]
\[ u[x_, t_] = \log[f[x, t]]; \]
Plot3D[u[x,t],{x,-15,15},{t,-15,15}]

u[x_]=2D[Log[f[x,-35]],x,x];
Plot[u[x],{x,-60,60},Filling→Bottom,PlotRange→{{-60,80},{0,0.7}}]

u[x_]=2D[Log[f[x,-15]],x,x];
Plot[u[x],{x,-60,60},Filling→Bottom,PlotRange→{{-60,80},{0,0.7}}]

u[x_]=2D[Log[f[x,30]],x,x];
Plot[u[x],{x,-60,60},Filling→Bottom,PlotRange→{{-60,80},{0,0.7}}]

shallow water wave equation:

k1=0.3;
f[x_,t_]=1+Exp[k1*x-k1/(1-k1^2)*t];
u[x_,t_]=2D[Log[f[x,t]],x,x];
Plot3D[u[x,t],{x,-20,20},{t,-20,20}]

u[x_]=2D[Log[f[x,-35]],x,x];
Plot[u[x],{x,-60,60},Filling→Bottom,PlotRange→{{-60,60},{0,0.3}}]

u[x_]=2D[Log[f[x,-15]],x,x];
Plot[u[x],{x,-60,60},Filling→Bottom,PlotRange→{{-60,60},{0,0.3}}]

u[x_]=2D[Log[f[x,20]],x,x];
Plot[u[x],{x,-60,60},Filling→Bottom,PlotRange→{{-60,60},{0,0.3}}]

k1=0.3;
k2=0.5;
a12=((k1^2-k1*k2+k2^2-3)*(k1-k2)^2)/((k1^2+k1*k2+k2^2-3)*(k1+k2)^2);
f[x_,t_]=1+Exp[k1*x-k1/(1-k1^2)*t]+Exp[k2*x-k2/(1-k2^2)*t]+a12*Exp[(k1+k2)*x-(k1/(1-k1^2)+k2/(1-k2^2))*t];
u[x_,t_]=2D[Log[f[x,t]],x,x];
Plot3D[u[x,t],{x,-15,15},{t,-15,15}]

u[x_]=2D[Log[f[x,-35]],x,x];
Plot[u[x],{x,-60,60},Filling→Bottom,PlotRange→{{-60,60},{0,0.3}}]
\[u[x_] = 2D[\log[f[x, -15]], x, x];\]
Plot\[u[x], \{x, -60, 60\}, Filling \to Bottom, PlotRange \to \{-60, 60\}, \{0, 0.3\}\]}

\[u[x_] = 2D[\log[f[x, 15]], x, x];\]
Plot\[u[x], \{x, -60, 60\}, Filling \to Bottom, PlotRange \to \{-60, 60\}, \{0, 0.3\}\]}

\[k1 = 0.3;\]
\[k2 = 0.5;\]
\[k3 = 0.7;\]
\[\theta_1 = k1*x - k1/(1 - k1^2)*t;\]
\[\theta_2 = k2*x - k2/(1 - k2^2)*t;\]
\[\theta_3 = k3*x - k3/(1 - k3^2)*t;\]
\[a12 = ((k1^2 - k1*k2 + k2^2 - 3)*(k1 - k2)^2)/((k1^2 + k1*k2 + k2^2 - 3)*(k1 + k2)^2);\]
\[a13 = ((k1^2 - k1*k3 + k3^2 - 3)*(k1 - k3)^2)/((k1^2 + k1*k3 + k3^2 - 3)*(k1 + k3)^2);\]
\[a23 = ((k2^2 - k2*k3 + k3^2 - 3)*(k2 - k3)^2)/((k2^2 + k2*k3 + k3^2 - 3)*(k2 + k3)^2);\]
\[b123 = a12*a13*a23;\]

\[f[x_, t_] = 1 + \exp[\theta_1] + \exp[\theta_2] + \exp[\theta_3] + a12*\exp[\theta_1 + \theta_2] + a23*\exp[\theta_2 + \theta_3] + a13*\exp[\theta_1 + \theta_3] + b123*\exp[\theta_1 + \theta_2 + \theta_3];\]

\[u[x_, t_] = 2D[\log[f[x, t]], x, x];\]
Plot\[3D[u[x, t], \{x, -15, 15\}, \{t, -15, 15\}\]}

\[u[x_] = 2D[\log[f[x, -35]], x, x];\]
Plot\[u[x], \{x, -60, 60\}, Filling \to Bottom, PlotRange \to \{-60, 80\}, \{0, 0.3\}\]}

\[u[x_] = 2D[\log[f[x, -15]], x, x];\]
Plot\[u[x], \{x, -60, 60\}, Filling \to Bottom, PlotRange \to \{-60, 80\}, \{0, 0.3\}\]}

\[u[x_] = 2D[\log[f[x, 20]], x, x];\]
Plot\[u[x], \{x, -60, 60\}, Filling \to Bottom, PlotRange \to \{-60, 80\}, \{0, 0.3\}\]}

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