Adomian Decomposition Method for Solving the Parabolic Equation

Mohammed Abd Alhameed Babiker Ahmed

B.Ed. in Mathematics and Islamic studies
University of the Holy Quran and Islamic Sciences (2011)

A Dissertation
Submitted to the University of Gezira in Partial Fulfillment of the Requirements for the Award of the Degree of Master of Science in

Mathematics

Department of Mathematics

Faculty of Mathematical and Computer Sciences

Feb, 2016
Adomian Decomposition Method for Solving the Parabolic Equation

Mohammed Abd Alhameed Babiker Ahmed

Supervision Committee:

Name: Dr. Abd Allah Habila Ali Kaitan
Position: Mani Supervisor
Signature: ………………

Name: Dr. Mohamed El-Neam Ahmed
Position: Co-supervisor
Signature: ………………

Date: Feb/ 2016
Adomian Decomposition Method for Solving the Parabolic Equation

Mohammed Abd Alhameed Babiker Ahmed

Examination Committee:

<table>
<thead>
<tr>
<th>Name</th>
<th>Position</th>
<th>Signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dr. Abd Allah Habila Ali Kaitan</td>
<td>Chair Person</td>
<td>..........</td>
</tr>
<tr>
<td>Dr. Mohammed Hassan Mohammed Khabir</td>
<td>External Examiner</td>
<td>..........</td>
</tr>
<tr>
<td>Dr. Murtada Khalafallah Elbashir</td>
<td>Internal Examiner</td>
<td>..........</td>
</tr>
</tbody>
</table>

Date of Examination: 22, Feb, 2016
Declaration

This to certify that I was completed my thesis by myself and using references without taking any information from others. This thesis was taken under supervision of advisors Dr. Abd Allah Habila Ali Kaitan.

Name: Mohammed Abd Alhameed Babiker Ahmed.
Dedication

Dedicated to my family who have spared no efforts to support me throughout the study.

I also dedicate this study to my lecturers who have provided me with full support and follow-up to accomplish this study.

To all persons who helped me through writing this thesis.
Acknowledgements

Thanks and praise exclusively is to Allah, the almighty and prayer and peace be upon our most truthful God's messenger. Thanks are also extended to those who never saved effort in offering me their good guidance, brilliant ideas and genuine advice. Comes foremost among them my supervisor Dr. Abd Allah Habila Ali Kaitan and Dr. Mohamed El-Neam Ahmed El-Nair Obied to whom I extend my wholehearted gratitude for his tolerance and patience. Also, thanks to the University staff that had helped me to build my future career.
Adomian Decomposition Method for Solving the Parabolic Equation
Mohammed Abd Alhameed Babiker Ahmed

ABSTRACT

The Adomian decomposition method (ADM) is a semi-analytical method for solving ordinary and partial nonlinear differential equations. The method was developed during the period 1970s - 1990s by George Adomian chair of the center of the applied mathematics at the University of Georgia. It is method further extensible to stochastic systems by using the Ito integral. The aim of this method is towards a unified theory for the more general theory of the homology analysis method. The important aspect of the method is the employment of “Adomian polynomial” which allows the convergence solution of the nonlinear portion of the equation. These polynomials mathematically generalize to a Maclaurin series about an arbitrary external parameter, which gives the solution method more flexibility than direct Taylor series expansion. The decomposition method and the modified decomposition method are related and effective techniques of promising results. Adomian method is well suited to solve Cauchy problems, an important class of problems which include initial condition problems. By using scientific mythology the Adomain decomposition method has been receiving much attention in several years in applied mathematics in general, and in the area of series solution in particular. In this research the first order linear partial differential equation were discussed. Homogeneous and nonhomogeneous, partial differential equations of first order are used to model traffic flow on a crowded road, blood flow through an elastic-walled tide shock waves, and special cases of the general theories of gas dynamics and hydraulics. A comparative study between the method of characteristics and the other two methods will be carried out through illustrative examples. In this study we show that the application of adomian decomposition method (ADM) to a class of partial differential equations provides a straightforward, accurate and quite efficient technique in comparison with the other usual classical methods. The results are also verified on two examples discussed at the end of the study. We recommend to apply the Adomian decomposition method for solving system of partial differential equations of higher orders.
حل معادلة القطع المكافئ بطريقة أدوميان التحليلية

محمد عبد الحميد يابكر أحمد

ملخص الدراسة

طريقه التحليل لأدوميان هي وسيلة شبه التحليلية لحل المعادلات التفاضلية العادية والجزئية تم تطوير الطرق من قبل جورج رئيس مركز الرياضيات التطبيقية في جامعة جورجيا في الفترة ما بين 1970 إلى 1990 تهدف هذه الطرق نحو نظرية مؤلفة والتي تسمح لأدوميان من أن تكون أعم من أساليب التحليل المتماثل العادي. علي جانب حاسم من هذه الطرق توظيف متعددة الحدود للنظام وتميمتها رياضيا. في صورة خطبة للنظر في الطرق وحل جزء غير الخطي من المعادلة نصل من غير سهولة إلى مستسلسة ماكلورين الذي يعطي طريقه الحل المحدد من المرونة من توسيع سلسلة تايلور مباشرة. وطرق أدومنيان التحليلية قابلة للتطوير والتحسين وربط الشروط مع الظاهر للوصول إلى نتائج مضبوطة. طريقة التحليلية لأدوميان تلتقي الكثير من الاهتمام في السنوات الأخيرة بالنسبة للرياضيات التطبيقية. واستخدمنا الطرق العلمية في هذه الدراسة واهتمام النتائج التي توصلت إليها الدراسة بالنسبة للمعادلات التفاضلية الجزئية الخطية من الدرجة الأولى كنموذج تدفق حركة المرور على طريق مزدحم وتدفق الدم من خلال مرونة الجريان والحالات الخاصة للمحظى. أما في دراسة مقارنة بين طريق الدرز والهيدروليكية والموجات والمصالح، وسيتم أيضاً تنفيذ دراسة مقارنة بين طريق الدرز وมาตรฐาน الأساليب من خلال أمثله توضيحية من فئة المعادلات التفاضلية الجزئية باستخدام طريقة أدومنيان. وفي هذه الدراسة أيضاً نوضح أن طريقه التحليل لأدومنيان توفر لنا تقنية واضحة ودقيقة وفعاله جداً بالمقارنة مع الطرق التقليدية المعتادة الأخرى. ويتحقق من هذا بإثبات من الامتثال التي نوقشت في نهاية الدراسة. وأوصي بأن تستخدم طريقه أدومنيان التحليلي في حل المعادلات التفاضلية الجزئية من الدرجة العليا.
# Table of Contents

<table>
<thead>
<tr>
<th>Contents Page</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>v</td>
</tr>
<tr>
<td>Abstract</td>
<td>vi</td>
</tr>
</tbody>
</table>

## Chapter One: Introduction to Adomian decomposition method

1.1 Introduction 1

1.2 Adomian decomposition method 1

1.3 Decomposition method 9

## Chapter Two: Problems on Adomian decomposition method

## Chapter Three: The modified decomposition method

3.1 Technique of method 26

3.2 Problems Use the modified decomposition method to solve the first order partial differential equation 27

## Chapter Four: Adomian decomposition method for solution of parabolic equation to nonlocal condition

4.1 Introduction 34

4.2 Analysis of the method (ADM) 34

4.3 Example 35

Example (4.1) we consider the parabolic problem (4.1),(4.3) 36

Example (4.2) we consider the parabolic problem (4.1),(4.3) 36

4.4 Conclusion and Recommendations 38

References 39
CHAPTER ONE

INTRODUCTION TO ADOMIAN DECOMPOSITION METHOD

1.1 Introduction:

In this chapter we will discuss the first order linear partial differential equations; homogeneous and inhomogeneous, partial differential equations of first order are used to model traffic flow on a crowded road, blood flow through elastic-walled tube shock waves, and as special cases of the general theories of gas dynamics and hydraulics. It is the concern of this text to introduce the recently developed methods handle partial differential equations in an accessible manner. Some of the traditional techniques will be used as well. In this thesis we will apply the Adomian decomposition method and the related phenomenon of the noise terms that will accelerate the rapid convergence of the solution. The decomposition method and the improvements made by the noise terms phenomenon and the modified decomposition method are related and effective techniques of promising results. Moreover the variation iteration method will be applied as will. These two methods provide the solution in an infinite series form. The obtained series may converge to a closed form solution if exact solution exists. For concrete problems where exact solution does not exist, the truncated series may be used for numerical purposes.

In addition to Adomian decomposition method and the variation iteration method, the classic method of characteristic will be used in this chapter. A comparative study between the method of characteristics and the other two methods will be carried out through illustrative examples.

1.2 Adomian Decomposition method:

In this section we will discuss the Adomian decomposition method. The Adomian decomposition method has been receiving much attention in applied mathematics in general, and in the area of series solutions in particular [12]. The method proved to be effective, powerful, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations. The decomposition method demonstrates fast convergence of the solution and therefore provides several significant advantages. In this thesis the method will be successfully used to handle most types of partial differential equations that appear in applied mathematics. We apply the method in a direct way without using linearization perturbation or any other
restrictive assumption that many change the physical behavior of the problems. The Adomian decomposition method was introduced and developed by George Adomian. This method is applied to a wide class of linear and nonlinear ordinary Partial differential equations and integral equations as well. The Adomian decomposition method consists of decomposing the unknown function \( u(x, y) \) of any equation in to a sum of an infinite number of components defined by the decomposition series:

\[
u(x, y) = \sum_{n=0}^{\infty} u_n(x, y),
\]

where the components \( u_n(x, y), n \geq 0 \) are to be determined in a continuously manner. The decomposition method concerns itself with finding the components \( u_0, u_1, u_2, ... \) one by one. Described here, the determination of these components can be achieved in an easy way through recursive relation that usually involves simple integrals. To give a clear overview of Adomian decomposition method, we have first consider the linear differential equation written in an operator by

\[
Lu + Ru = g,
\]

where \( L \) is, mostly, the lower order derivative which is assumed to invertible, \( R \) is other linear differential operator, and \( g \) is a source term. We next apply the inverse operator \( L^{-1} \) to both sides of equation (1.2) we get

\[
u = f - L^{-1}(Ru),
\]

where the function represents the terms arriving from integrating the source term \( g \) and from using the given condition that are assumed to be prescribed. As indicated before, Adomian method defines the solution \( u \) by an infinite series of components given by

\[
u = \sum_{n=0}^{\infty} u_n,
\]

where the components \( u_0, u_1, u_2, ... \) are usually recurrently determined. Substituting in equation (1.4) in to both sides of equation (1.3) therefore,

\[
\sum_{n=0}^{\infty} u_n = f - L^{-1}(R(\sum_{n=0}^{\infty} u_n)),
\]

For simplicity equation (1.5) can be rewritten as:

\[
u_0 + u_1 + u_2 + ... = f - L^{-1}(R(u_0 + u_1 + u_2 + ...)),
\]
To construct the recursive relation needed for the determination of the components $u_0, u_1, u_2, \ldots$, it is important to note that Adomain method suggests that the zeroth component $u_0$ is usually defined by the function $f$ described above, i.e. by all terms, that are not included under the inverse operator $L^{-1}$, that arise from integrating the inhomogeneous term and from the initial data. Accordingly, the formal continuous relation is defined by $u_0 = f$, therefore,

$$u_{k+1} = -L(R(u_k)), k \geq 0. \quad (1.7)$$

Or equivalently

$$u_0 = f,$$

$$u_1 = -L^{-1}(R(u_0)),$$

$$u_2 = -L(R(u_1)),$$

$$u_3 = -L^{-1}(R(u_2)). \quad (1.8)$$

It is clearly seen that the relation (1.8) reduced the differential equation under consideration in to an elegant determination of computable components. Substitute equation (1.8) into equation (1.4) to obtain the solution in a series form. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converge very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of resulting series. Cherruault examind the convergence of Adomians method. Moreover, Cherruault and Adomian presented a new proof of convergence of the method. For more details about the proof presented to discuss the rapid convergence. However, for concrete problems, where a closed from solution is not obtainable, a truncated number of terms is usually used for numerical purposes. It was also shown by many that the series obtained by evolution few terms given an approximation of high degree of accuracy where compared with other numerical techniques.

It seems reasonable to give a brief outline about the works conducted by Adomian and other researchers in applying Adomian's method. Adomian in equation (1.2) and in many other works introduced his method and applied it to many stochastic and deterministic problems. He implemented his method to solve frontier
problems of physics. The Adomian's achievements there, are remarkable and of promising results.

A comparison between the decomposition method and the perturbation technique showed the efficiency of the decomposition method compared to the much work required by the perturbation method. Also a comparative study between Adomian's method and Taylor series method requires less computational work if compared with Taylor series method has been examined to show that the decomposition method requires less computational work if compared with Taylor series. The advantage of the decomposition method over Picard’s methods has been emphasized in many works. Other comparison with traditional method such as finite difference method have been conducted the literature.

It is to be noted that many studies shown that few terms of the decomposition series provide a numerical result of a high degree of accuracy. Employed Adomian's method to solve differential equations with singular coefficients such as Legendre’s equations, Bessel's equations, and Hermit equation, moreover ,a suitable definition of the operator was used to overcome the difficulty of singular points of lane-Emden equation, a new definition of the operator was introduced to overcome the singularity for the lane-Emden Type of equations. Many other studies implement the decomposition method for differential equations, ordinary and partial, and for integral equations, linear and nonlinear (see [12],[13]).

It is normal in differential equations that we seek a closed form solution or a series solution with a proper number of terms. Although this thesis is devoted to handle partial differential equation, but it seems reasonable to use the decomposition method to discuss two ordinary deferential equations where an exact solution is obtained for the first equation and a series approximation is determined for the second equation. For the first problem we consider the equation:

\[ u'(x) = u(x), u(0) = A. \]  

In an operator form equation (1.9) becomes

\[ Lu = u, \]  

where the differential operator \( L \) is given by

\[ L = d/dx, \]  

and therefore the inverse operator \( L^{-1} \) is defined by
\[ L^{-1}(.) = \int_{0}^{x}(.)(\cdot)dx. \quad (1.12) \]

Applying \( L^{-1} \) to both sides of equation (1.10) and using the initial condition we have

\[ L^{-1}(Lu) = L^{-1}(u), \quad (1.13) \]

So that

\[ u(x) - u(0) = L^{-1}(u), \quad (1.14) \]

Or equivalently

\[ u(x) = A + L^{-1}(u), \quad (1.15) \]

Substituting the series assumption in equation (1.14) in to side equation (1.15) gives

\[ \sum_{n=0}^{\infty} u_n(x) = A + L^{-1}\left(\sum_{n=0}^{\infty} u_n(x)\right), \quad (1.16) \]

In view of equation (1.16) the following continuous relation \( u_0(x) = A \), we obtain,

\[ u_{k+1}(x) = L^{-1}(u_k(x)), k \geq 0. \quad (1.17) \]

Follows immediately, consequently, we obtain \( u_0(x) = A \), we get,

\[ u_1(x) = L^{-1}(u_0(x)) = Ax, \]
\[ u_2(x) = L^{-1}(u_0(x)) = A \frac{x^2}{2!}, \quad (1.18) \]
\[ u_3(x) = L^{-1}(u_2(x)) = A \frac{x^3}{3!}, \]

Substituting equation (1.18) into equation (1.17) gives the solution in series form, given by:

\[ u(x) = A(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...) \quad (1.19) \]

And in a closed form by

\[ u(x) = Ae^x. \quad (1.20) \]

We next consider the well-known Airy’s equation we get,

\[ u''(x) = xu(x), u(0) = A, u'(0) = B, \quad (1.21) \]

In an operator form equation (1.21) becomes

\[ Lu = xu, \quad (1.22) \]
Where the differential operator $L$ is given by

$$L = \frac{d^2}{dx^2}.$$  \hfill (1.23)

And therefore inverse operator $L^{-1}$ is defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot) dx \, dx \hfill (1.24)$$

Operating with $L^{-1}$ on both sides of equation (1.22) and using the initial conditions we get

$$L^{-1}(Lu) = L^{-1}(xu), \hfill (1.25)$$

So that

$$u(x) - xu'(0) - u(0) = L^{-1}(xu), \hfill (1.26)$$

Or equivalently

$$u(x) = A + Bx + L^{-1}(xu), \hfill (1.27)$$

Substituting the series assumption (1.5) into both sides yields

$$\sum_{n=0}^{\infty} u_n(x) = A + Bx + L^{-1}(xu_k(x)), K \geq 0, \hfill (1.28)$$

Following the decomposition method we obtain the following continuous relation

$$u_0(x) = A + Bx, \hfill (1.29)$$

Consequently, we get

$$u_0(x) = A + Bx,$$

$$u_1(x) = L^{-1}(xu_0(x)) = A \frac{x^3}{6} + B \frac{x^4}{12}, \hfill (1.30)$$

$$u_2(x) = L^{-1}(xu_1(x)) = A \frac{x^6}{180} + B \frac{x^7}{504},$$

Substituting in equation (1.31) into equation (1.5) gives the solution in a series form, given by:

$$u(x) = A \left( 1 + \frac{x^3}{6} + \frac{x^6}{180} + \ldots \right) + B \left( x + \frac{x^4}{12} + \frac{x^7}{504} + \ldots \right). \hfill (1.31)$$

The accuracy of the approximation can be easily computed for other components. It seems now reasonable to apply Adomian decomposition method to first-order partial differential equations. To explain that, and without loss of generality, we consider the inhomogeneous partial differential equation:
\( u_x + u_y = f(x, y), u(0, y) = g(y), u(x, 0) = h(x), \) \hspace{1cm} (1.32)

Equation (1.32) in an operator form can be written as

\[
L_x u + L_y u = f(x, y),
\]

where

\[
L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y},
\]

(1.34)

where each operator is assumed easily invertible, and thus the inverse operators \( L_x^{-1} \) and \( L_y^{-1} \) exist and given by

\[
L_x^{-1}(. ) = \int_0^x (.) dx,
\]

(1.35)

\[
L_y^{-1}(. ) = \int_0^y (.) dy,
\]

(1.36)

Applying \( L_x^{-1} \) to both sides of equation (1.34) gives

\[
L_x^{-1} L_x u = L_x^{-1}(f(x, y)) - L_x^{-1}(L_y u),
\]

(1.37)

or equivalently

\[
u(x, y) = g(y) + L_x^{-1}(f(x, y)) - L_x^{-1}(L_y u),
\]

(1.38)

Obtained by using equation (1.36) and by using the condition \( u(0, y) = g(y) \). As stated above the decomposition method stets.

\[
u(x, y) = \sum_{n=0}^{\infty} u_n(x, y),
\]

(1.39)

Substituting in equation (1.39) in to both sides of equation (1.39) we get

\[
\sum_{n=0}^{\infty} u_n(x, y) = g(y) + L_x^{-1}(f(x, y)) - L_x^{-1} \left( L_y \sum_{n=0}^{\infty} u_n(x, y) \right),
\]

(1.40)

This can be rewritten as

\[
u_0 + u_1 + u_2 + \ldots = g(y) + L_x^{-1}(f(x, y)) - L_x^{-1} L_y (u_0 + u_1 + u_2 + \ldots),
\]

(1.41)

As suggested by Adomian method, the zeros component \( u_0 \), is always identified by the given initial condition and the terms arriving form \( L_x^{-1}(f(x, y)) \), both of which are assumed to be known, accordingly we set

\[
u_0(x, y) = g(y) + L_x^{-1}(f(x, y)),
\]

(1.42)
Consequently, the other components $u_{k+1}, k \geq 0$, are defined by using the relation

$$u_{k+1}(x, y) = -L_x^{-1} L_y^{-1} (u_k), k \geq 0.$$  \hspace{0.5cm} (1.43)

Combining, equation (1.42) and equation (1.43) we obtain the continuous scheme

$$u_0(x, y) = g(y) + L_x^{-1} (f(x, y)), \hspace{0.5cm} (1.44)$$

$$u_{k+1}(x, y) = -L_x^{-1} L_y^{-1} (u_k), k \geq 0.$$  \hspace{0.5cm} (1.45)

That forms basis for a complete determination of the components $u_0, u_1, u_2, ...$ therefore, we can easily obtain the components given by:

$$u_0(x, y) = g(y) + L_x^{-1} (f(x, y)), \hspace{0.5cm} (1.46)$$

$$u_1(x, y) = -L_x^{-1} (Ly u_0(x, y)), \hspace{0.5cm} (1.47)$$

$$u_2(x, y) = -L_x^{-1} (Ly u_1(x, y)), \hspace{0.5cm} (1.48)$$

$$u_3(x, y) = -L_x^{-1} (Ly u_2(x, y)), \hspace{0.5cm} (1.49)$$

And so on. Thus the components $u_n$ can be determined continuous as far as we like. It is clear that the accuracy of the approximation can be significantly improved by simply determining more components. Having established the components of $u(x, y)$ the solution in a series form follows immediately. However the expression

$$\phi_n = \sum_{r=0}^{n-1} u_r (x, y), \hspace{0.5cm} (1.50)$$

Is considered the n-term approximation to $u$, for concrete problems, where exact solution is not easily obtainable purposes. As mentioned earlier, the convergence of Adomian decomposition method has been established by many researchers, but will not be discussed in this thesis.

It is important to note that the solution can also be obtained by finding the y-solution by applying the inverse operator $L_y^{-1}$ to both sides of the equation

$$L_y = f(x, y) - L_x u.$$  \hspace{0.5cm} (1.51)

The equality of the x-solution and the y-solution is formally justified and will be examined through the coming examples.

It should be noted here that the series solution equation (1.39) has been proves by many researchers to converge rapidly and a closed form solution is obtainable in many cases if a closed form solution exist. It was found, as will be seen later, that
very few terms of the series obtained in equation (1.39) provide a high degree of accuracy level which makes the method powerful when compared with other existing numerical techniques. In many case the series representation of \( u(x, y) \) can be summed to yield the closed form solution. There are many works in this direction have demonstrated the power of the method for analytical and numerical applications.

1.3 **Decomposition method**: 

We can make an outline about the essential feature of the decomposition method for linear and nonlinear equations homogeneous and inhomogeneous as follows:

1. Express the partial differential equation, linear or nonlinear in an operator form.
2. Apply the inverse operator to both sides of the equation written in an operator form.
3. Set the unknown function \( u(x, y) \) into a decomposition series.

\[
\sum_{n=0}^{\infty} u_n(x, y),
\]

Whose components are elegantly determined. We next substitute the series equation (1.48) into both sides of the resulting equation.

4. Identify the zeros component \( u_0(x, y) \) as the terms arising form integrating the source term and from the given conditions, \( f(x, y) \), both are assumed to be known.

5. Determine the successive components of the series solution \( u_k, k \geq 1 \) by applying the continuous scheme equation (1.44) where each component \( u_k \) can be completely determined that, by using the previous component \( u_{k-1} \).

6. Substitute the determined components equation (1.48) to obtain the solution in a series form. An exact solution can be easily obtained in many equations of such a closed form solution exists.

It is to be noted that Adomian decomposition method approaches any equation linear or nonlinear, and homogeneous or inhomogeneous in straightforward manner without any need to restrictive assumptions such as linearization, discrimination or perturbation. There is no need in using this method to convert inhomogeneous conditions to homogeneous conditions as required by other techniques [12].
CHAPTER TWO
PROBLEMS ON ADOMIAN DECOMPOSITION METHOD

2.1 Use Adomian decomposition method to solve the following inhomogeneous PDE:

\[ u_x + u_y = x^2 + y^2, \quad u(0, y) = 0, \quad u(x, 0) = 0, \quad (2.1) \]

Solution:

In an operator form Eq. (2.1) can be written as

\[ L_x u = x^2 + y^2 - L_y u \quad (2.2) \]

Where

\[ L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}, \quad (2.3) \]

It is clear that \( L_x \) is invertible, hence \( L_x^{-1} \) exists and given by

\[ L_x^{-1} (.) = \int_{0}^{x} (.) dx, \quad (2.4) \]

The x– solution:

This solution can be obtained by applying \( L_x^{-1} \) to both sides of equation (2.2) hence we find

\[ L_x^{-1} L_x u = L_x^{-1} (x^2 + y^2) - L_x^{-1} (L_y u), \quad (2.5) \]

Or equivalently

\[ u(x, y) = u(0, y) + \frac{1}{3} x^3 + x y^2 - L_x^{-1} (L_y u) = \frac{1}{3} x^3 + x y^2 - L_x^{-1} (L_y u), \quad (2.6) \]

Obtained upon using the given condition \( u(0, y) \), equation (2.2) and by integrating \( f(x, y) = x + y \) with respect to \( x \), As stated above the decomposition method identifies unknown function \( u(x, y), n \geq 0 \) as an infinite number of components \( u_n(x, y), n \geq 0 \) given by

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y), \quad (2.7) \]

Substituting (2.7) in to both sides of (2.7) we find

\[ \sum_{n=0}^{\infty} u_n(x, y) = \frac{1}{3} x^3 + x y^2 - L_x^{-1} (L_x \left( \sum_{n=0}^{\infty} u_n(x, y) \right)), \quad (2.8) \]
Using few term of the decomposition (2.7) we obtain

\[ u_0 + u_1 + u_2 + \ldots = \frac{1}{3} x^3 + xy^2 - L_y (u_0 + u_1 + u_2 + \ldots), \]  
(2.9)

As presented before, the decomposition method identifies the zeroth component \( u_0 \) by all terms arising from the given condition and from integrating \( f(x, y) = x + y \) therefore we set

\[ u_0 (x, y) = \frac{1}{3} x^3 + xy^2, \]  
(2.10)

Consequently, the recursive scheme that will enable us to completely determine the successive components is thus constructed by

\[ u_0 (x, y) = \frac{1}{3} x^3 + xy^2, \]  
(2.11)

\[ u_{k+1} (x, y) = -L_y^{-1} \left( L_y \left( u_k \right) \right), \quad k \geq 0. \]

This in turn gives

\[ u_1 (x, y) = -L_y^{-1} \left( L_y u_0 \right) = -L_y^{-1} \left( L_y \left( \frac{1}{3} x^3 + xy^2 \right) \right) = -x^2 y, \]  
(2.12)

\[ u_2 (x, y) = -L_y^{-1} \left( L_y \left( u_1 \right) \right) = -L_y^{-1} \left( L_y \left( -x^2 y \right) \right) = \frac{1}{3} x^3. \]

Accordingly, \( u_k = 0, k \geq 0 \). Having determined the components \( u(x, y) \), we find

\[ u(x, y) = u_0 + u_1 + u_2 + \ldots = \frac{1}{3} x^3 + xy^2 - x^2 y + \frac{1}{3} x^3. \]  
(2.13)

The exact solution of the equation under discussion.

**The y – solution:**

It is important to note that the exact solution can also be obtained by finding the \( y \)-solution. In an operator form we can write the equation by

\[ L_y = x^2 + y^2 - L_x u, \]  
(2.14)

Assume that \( L_y^{-1} \) exists and defined by

\[ L_y^{-1} (.) = \int_0^y (.) dy, \]  
(2.15)

Applying \( L_y^{-1} \) to both sides of the equation (2.14) gives

\[ u(x, y) = x^2 y + \frac{1}{3} y^3 - L_y^{-1} \left( L_x u \right), \]  
(2.16)
As mentioned above, the decomposition method sets the solution \( u(x,y) \) in a series form by

\[
  u(x,y) = \sum_{n=0}^{\infty} u_n(x,y),
\]

(2.17)

Inserting in equation (2.17) in to both sides of equation (2.16) we obtain

\[
  \sum_{n=0}^{\infty} u_n(x,y) = x^2 y + \frac{1}{3} y^3 - L_y \left( L_x \left( \sum_{n=0}^{\infty} u_n(x,y) \right) \right),
\]

(2.18)

Using few terms only for simplicity reasons we obtain

\[
  u = u_0 + u_1 + u_2 + \cdots = x^2 y + \frac{1}{3} y^3 - L_y \left( L_x (u_0 + u_1 + u_2 + \cdots) \right),
\]

(2.19)

The decomposition method identifies the zeroth component \( u_0 \) by all terms arising from the given condition and from integrating \( f(x,y) = x + y \) therefore we set

\[
  u_0(x,y) = x^2 y + \frac{1}{3} y^3,
\]

(2.20)

To completely determine the successive components of \( u(x,y) \), the recursive scheme is thus defined by

\[
  u_0(x,y) = x^2 y + \frac{1}{3} y^3,
\]

(2.21)

\[
  u_{k+1}(x,y) = -L_y \left( L_x (u_k) \right), k \geq 0.
\]

This gives

\[
  u_1(x,y) = -L_y \left( L_x (u_0) \right) = -L_y \left( L_x \left( x^2 y + \frac{1}{3} y^3 \right) \right) = -x y^2,
\]

(2.22)

\[
  u_2(x,y) = -L_y \left( L_x (u_1) \right) = -L_y \left( L_x \left( -x y^2 \right) \right) = \frac{1}{3} y^3.
\]

Consequently \( u_k = 0, k \geq 2 \). Having determined the components of \( u(x,y) \) we find

\[
  u(x,y) = u_0 + u_1 + u_2 + \cdots = x^2 y + \frac{1}{3} y^3 - x y^2 + \frac{1}{3} y^3.
\]

(2.23)

The exact solution of the equation under discussion.
2.2 Solve the following homogenous partial differential equation:

\[ u_x + u_y = 0, \quad (2.24) \]

\[ u(0, y) = y, \quad u(x, 0) = x, \]

**Y-Solution:**

In an operator form equation (2.24) becomes

\[ L_x u(x, y) = -L_y u(x, y), \quad (2.25) \]

Where the operators \( L_x \) and \( L_y \) are defined by

\[ L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}. \quad (2.26) \]

Applying the inverse operator \( L_x^{-1} \) to both sides equation (2.25) and using the given condition \( u(0, y) = y \) yields

\[ u(x, y) = x - L_y^{-1}(L_x u), \quad (2.27) \]

We next define the unknown function \( u(x, y) \) by the decomposition series

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y), \quad (2.28) \]

Interesting in equation (2.28) into both sides of equation (2.27) gives

\[ \sum_{n=0}^{\infty} u_n(x, y) = x - L_y^{-1}\left( L_x \left( \sum_{n=0}^{\infty} u_n(x, y) \right) \right). \quad (2.29) \]

By considering few terms of the decomposition of \( u(x, y) \), equation (2.29) becomes

\[ u_0 + u_1 + u_2 + \ldots = x - L_y^{-1}(L_x(u_0 + u_1 + u_2 + \ldots)), \quad (2.30) \]

Proceeding as before we identify the zeros components \( u_0 \) by

\[ u_0(x, y) = x, \quad (2.31) \]

Having identified the zeros component \( u_0(x, y) \) we obtain the recursive scheme

\[ u_0(x, y) = x, \quad (2.32) \]

\[ u_{k+1}(x, y) = -L_y^{-1}L_x(u_k), \quad k \geq 0, \]

The components \( u_0, u_1, u_2, \ldots \) are thus determined as follows:

\[ u_0(x, y) = x, \]

\[ u_1(x, y) = -L_y^{-1}L_x(u_0) = -L_y^{-1}L_x(x) = -y, \quad (2.33) \]

\[ u_2(x, y) = -L_y^{-1}L_x(u_1) = -L_y^{-1}L_x(-y) = 0, \]
It is obvious that all components \( u_k(x, y) = 0, k \geq 2 \) consequently the solution is given by

\[
u(x, y) = u_0(x, y) + u_1(x, y) + \ldots = u_0(x, y) + u_1(x, y) = x - y.
\] (2.34)

The exact solution obtained by using the decomposition series equation (2.28).

It is important to note here that the exact solution given by equation (2.34) can also be obtained by determining the \( y \)-solution as discussed above.

2.3 **Solve the following homogeneous partial differential equation:**

\[
x u_x + u_y = 3u, u(x,0) = x^2, u(0, y) = 0,
\] (2.35)

**Solution:**

In an operator form equation (2.35) becomes

\[
L_y u(x, y) = 3u(x, y) - xL_x u(x, y),
\] (2.36)

Applying the inverse operator \( L_y^{-1} \) to both sides of equation (2.36) and using the given condition \( u(x,0) = x^2 \) yields

\[
u(x, y) = x^2 + L_y^{-1}(3u - xL_x u),
\] (2.37)

Substituting \( u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \) into both sides of equation (2.37) gives

\[
\sum_{n=0}^{\infty} u_n(x, y) = x^2 + L_y^{-1}\left[ 3\sum_{n=0}^{\infty} u_n(x, y) - xL_x \sum_{n=0}^{\infty} u_n(x, y) \right],
\] (2.38)

By considering few terms of the decomposition of \( u(x, y) \), equation (2.38) becomes

\[
u_0 + u_1 + u_2 + \ldots = x^2 + L_y^{-1}(3u_0 + u_1 + \ldots) - xL_x (u_0 + u_1 + u_2 + \ldots),
\] (2.39)

proceeding as before, we identify the recursive scheme.

\[
u_0(x, y) = x^2,
\]

(4.40)

\[
u_{k+1}(x, y) = L_y^{-1}(3u_k - xL_x u_k), k \geq 0.
\]

The components \( u_0, u_1, u_2, \ldots \) are thus determined as follows:

\[
u_0(x, y) = x^2,
\]

\[
u_1(x, y) = L_y^{-1}(3u_0 - xL_x u_0) = x^2 y,
\]

\[
u_2(x, y) = L_y^{-1}(3u_1 - xL_x u_1) = \frac{x^2 y^2}{3!},
\] (2.41)
\[ u_3(x, y) = L_x^{-1}(3u_2 - xL_x u_2) = \frac{x^2 y^3}{3!}, \]

Consequently the solution is given by

\[ u(x, y) = u_0 + u_1 + u_2 + ... = x^2(1 + y + \frac{y^2}{2!} + ...) = x^2 e^y. \tag{2.42} \]

**2.4 Solve the following homogeneous PDE:**

**2.4.1** \[ u_x - yu = 0, \quad u(0, y) = 1, \tag{2.43} \]

**Solution:**

In an operator form equation (2.43) becomes

\[ L_x u(x, y) = yu(x, y), \tag{2.44} \]

where the operator \( L_x \) is defined as

\[ L_x = \frac{\partial}{\partial x}. \tag{2.45} \]

Applying the integral operator \( L_x^{-1} \) to both sides of equation (2.44) and using the given condition that \( u(0, y) = 1 \), gives

\[ u(x, y) = 1 + L_x^{-1}(yu(x, y)), \tag{2.46} \]

following the discussion presented above we define the unknown function \( u(x, y) \) by the decomposition series

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y), \tag{2.47} \]

interesting in equation (2.47) into sides of equation (2.46) gives

\[ \sum_{n=0}^{\infty} u_n(x, y) = 1 + \left( y \sum_{n=0}^{\infty} u_n(x, y) \right), \tag{2.48} \]

or equivalently

\[ u_0 + u_1 + u_2 + ... = 1 + L_x^{-1}\left(y(u_0 + u_1 + u_2 + ...\right)), \tag{2.49} \]

by considering few terms of the decomposition of \( u(x, y) \) the components \( u_0, u_1, u_2, ... \) are thus determined by using the recursive relationship as follows:

\[ u_0(x, y) = 1, \]
\[ u_1(x, y) = L_x^{-1}(yu_0) = xy, \]
\[ u_2(x, y) = L_x^{-1}(yu_1) = \frac{1}{2!} x^2 y^2, \tag{2.50} \]
\[ u_y(x, y) = L_x^{-1}(y u_x) = \frac{1}{3!} x^3 y^3, \]

and so on for other components. Consequently, the solution in series form is given by

\[ u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) + \ldots, \]

\[ = 1 + xy + \frac{1}{2!} x^2 y^2 + \frac{1}{3!} x^3 y^3 + \ldots, \quad (2.51) \]

And in a closed form

\[ u(x, y) = e^{xy}. \quad (2.52) \]

**2.4.2** \[ u_t + cu_x = 0, \quad u(x, 0) = x, \quad (2.53) \]

where \( c \) is constant.

**Solution:**

In an operator form equation (2.53) can be rewritten as

\[ L_t u(x, t) = -cL_x u, \quad (2.54) \]

where the operator \( L_t \) is defined as

\[ L_t = \frac{\partial}{\partial t}, \quad (2.55) \]

It is clear that the operator \( L_t \) is invertible, and the inverse operator \( L_t^{-1} \) is an indefinite integral form o to t. Applying the integral operator \( L_t^{-1} \) to both sides of equation (2.54) and using the given condition that \( u(x, 0) = x \) yields

\[ u(x, t) = x - cL_t^{-1}(L_x u(x, t)), \quad (2.56) \]

Proceeding as before, we substitute the decomposition series for in to sides of (2.56) to obtain

\[ \sum_{n=0}^{\infty} u_n(x, t) = x - cL_t^{-1}\left( L_x \left( \sum_{n=0}^{\infty} u_n(x, t) \right) \right), \quad (2.57) \]

Using few terms of the decomposition of \( u(x, y) \), equation (2.57) becomes

\[ u_0 + u_1 + u_2 + \ldots = x - cL_t^{-1}\left( L_x (u_0 + u_1 + u_2 + \ldots) \right), \quad (2.58) \]

The components \( u_0, u_1, u_2, \ldots \) can be determined by using the recursive relationship as follows:
\[ u_0(x, t) = x, \]
\[ u_1(x, t) = -cL^{-1}_x(L_xu_0) = -ct, \]
\[ u_2(x, t) = -L^{-1}_x(L_xu_1) = 0. \tag{2.59} \]

We can easily observe that \( u_k = 0, k \geq 2 \). It follows that the solution in a closed form is given by

\[ u(x, t) = x - ct. \]

### 2.5 Solve the following partial differential equation:

\[
\begin{align*}
  u_x + u_y + u_z &= u, u (0, y, z) = 1 + e^x + e^z, \\
  u(x, 0, z) &= 1 + e^x + e^z, u(x, y, 0) = 1 + e^x + e^y,
\end{align*}
\tag{2.61}
\]

where \( u = u(x, y, z) \)

#### Solution:

In an operate from equation (2.61) can be rewritten as

\[ L_x + u(x, y, z) = u - L_xu - L_yu, \tag{2.62} \]

where the operator \( L_x, L_y \) and \( L_z \) are defined by

\[ L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}, \quad L_z = \frac{\partial}{\partial z}. \tag{2.63} \]

Assume that the operator \( L_x \) is invertible, and the inverse operator \( L_x^{-1} \) is an indefinite integral from 0 to \( x \). Applying the integral operator \( L_x^{-1} \) to both sides of equation (2.62) and using the given condition that \( u(0, y, z) = 1 + e^y + e^z \) yields

\[ u(x, y, z) = 1 + e^x + e^z + L_x^{-1}(u - L_yu - L_zu), \tag{2.64} \]

Proceeding as before, we substitute the decomposition

\[ u(x, y, z) = \sum_{n=0}^{\infty} u_n(x, y, z), \tag{2.65} \]

In to both sides of equation (2.64) to find

\[ \sum_{n=0}^{\infty} u_n(x, y, z) = 1 + e^x + e^z + L_x^{-1}\left(\sum_{n=0}^{\infty} u_n - L_y\left(\sum_{n=0}^{\infty} u_n\right) - L_z\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{2.66} \]

Using few terms of the decomposition of \( u(x, y, z) \), equation (2.66) becomes

\[ u_0 + u_1 + u_2 + \ldots = 1 + e^x + e^z + L_x^{-1}(u_0 + u_1 + u_2 + \ldots) - L_y\left(L_x(u_0 + u_1 + u_2 + \ldots)\right) - L_z\left(L_x(u_0 + u_1 + u_2 + \ldots)\right), \tag{2.67} \]
The components $u_0 + u_1 + u_2 + ...$ can be determined recurrently as follows

$$u_0(x, y, z) = 1 + e^y + e^z,$$
$$u_1(x, y, z) = L_x^1(u_0 - L_x u_0 - L_x u_0) = x,$$
$$u_2(x, y, z) = L_x^1(u_1 - L_x u_1 - L_x u_1) = \frac{1}{2!}x^2,$$
$$u_3(x, y, z) = L_x^1(u_2 - L_x u_2 - L_x u_2) = \frac{1}{3!}x^3,$$

and so on consequently, the solution in a series form is given by

$$u(x, y, z) = (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...) + e^y + e^z,$$

(2.69)

and in closed form

$$u(x, y, z) = e^y + e^z + e^z.$$

(2.70)

2.6 Solve the following PDE:

$$u_x + u_y = 2xy^2 + 2x^2y, \quad u(x,0) = 0, u(0,y) = 0,$$

(2.71)

Solution:

In an operator from above equation (2.71) can be written as

$$L_x u = 2xy^2 + 2x^2y - L_y u,$$

(2.72)

where

$$L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}.$$

(2.73)

It is clear that invertible hence $L_x^{-1}$ exists and given by

$$L_x^{-1}(. \ ) = \int_0^x (. \ )dx$$

(2.74)

The x – solution:

The solution can be obtained by applying $L_x^{-1}$ to both sides of equation (2.71) hence we find

$$L_x^{-1} L_x u = L_x^{-1} (2xy^2 + 2x^2y) - L_x^{-1} (L_y u),$$

(2.75)

Or equivalently

$$u(x, y) = u(0, y) + x^2 y^2 + \frac{2}{3}x^3 y - L_x^{-1}(L_y u),$$

(2.76)
\[ u(0, y) = 0 \]

\[ \Rightarrow u(x, y) = x^2 y^2 + \frac{2}{3} x^3 y - L_x^{-1}(L_y u), \quad (2.77) \]

As stated above the decomposition method identifies the unknown function \( u(x, y) \) as an infinite number of components \( u_n(x, y), n \geq 0 \) given by

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y), \quad (2.78) \]

Substituting in equation (2.77) in to both sides of equation (2.76) we find

\[ \sum_{n=0}^{\infty} u_n(x, y) = x^2 y^2 + \frac{2}{3} x^3 y - L_x^{-1}\left(L_y \sum_{n=0}^{\infty} u_n(x, y)\right), \quad (2.79) \]

Using few terms of decomposition equation (2.77) we obtain

\[ u_0 + u_1 + u_2 + \ldots = x^2 y^2 + \frac{2}{3} x^3 y - L_x^{-1}\left(L_y (u_0 + u_1 + u_2 + \ldots)\right), \quad (2.80) \]

\[ u_0(x, y) = x^2 y^2 + \frac{2}{3} x^3 y, \quad (2.81) \]

\[ u_{k+1}(x, y) = -L_x^{-1}(L_y (u_k)), k \geq 0. \]

\[ u_1(x, y) = -L_x^{-1}(L_y (u_0)) = -L_x^{-1}\left(L_y \left( x^2 y^2 + \frac{2}{3} x^3 y \right) \right) = -L_x^{-1}\left(2x^2 y + \frac{2}{3} x^3 \right) \]

\[ \Rightarrow u_1(x, y) = -\frac{2}{3} x^3 y - \frac{2}{12} x^4. \quad (2.82) \]

\[ u_2(x, y) = -L_x^{-1}(L_y (u_1)) = -L_x^{-1}\left(L_y \left( \frac{2}{3} x^3 y - \frac{2}{12} x^4 \right) \right) = -L_x^{-1}\left(-\frac{2}{3} x^3 \right). \]

\[ \Rightarrow u_2(x, y) = \frac{2}{12} x^4. \quad (2.83) \]

Accordingly \( u_k = 0, k \geq 0 \) Having determined the components of \( u(x, y) \) we find

\[ u = u_0 + u_1 + u_2 + \ldots = x^2 y^2 + \frac{2}{3} x^3 y - \frac{2}{3} x^3 y - \frac{2}{12} x^4 + \frac{2}{12} x^4 = x^2 y^2, \quad (2.84) \]

The exact solution of the equation under discussion.
The y – solution:

we can write the equation by

\[ L_y u = 2x y^2 + 2x^2 y - L_x u, \]  

(2.85)

Assume that \( L_y^{-1} \) exists and by

\[ L_y^{-1} (.) = \int_0^y (. ) d y \]  

(2.86)

Applying \( L_y^{-1} \) to both sides if the equation (2.85)

\[ L_y^{-1} L_y = L_y^{-1} (2xy^2 + 2x^2 y) - L_y^{-1} L_x (u), \]  

(2.87)

\[ u(x, y) = \frac{2}{3} xy^3 + x^2 y^2 - L_y^{-1} L_x (u), \]  

(2.88)

As mentioned above the decomposition sets the solution \( u(x, y) \) in a series form by

\[ u(x, y) = \sum_{n=0}^{\infty} u_n (x, y), \]  

(2.89)

Interested in equation (2.88) in to both sides equation (2.89)

\[ \sum_{n=0}^{\infty} u_n (x, y) = \frac{2}{3} xy^3 + x^2 y^2 - L_y^{-1} \left( L_x \left( \sum_{n=0}^{\infty} u_n (x, y) \right) \right), \]  

(2.90)

Using few terms only for simplicity reasons we obtain

\[ u_0 + u_1 + u_2 + \ldots = \frac{2}{3} xy^3 + x^2 y^2 - L_y^{-1} \left( L_x (u_0 + u_1 + u_2 + \ldots) \right), \]  

(2.91)

\[ u_0 (x, y) = \frac{2}{3} xy^3 + x^2 y^2, \]  

(2.92)

\[ u_{k+1} (x, y) = -L_y^{-1} \left( L_x (u_k) \right), k \geq 0. \]

This gives

\[ u_1 (x, y) = -L_y^{-1} \left( L_x (u_0) \right) = -L_y^{-1} \left( L_x \left( \frac{2}{3} xy^3 + x^2 y^2 \right) \right) \]

\[ = -L_y^{-1} \left( \frac{2}{3} xy^3 + 2xy^2 \right) \]

\[ \therefore \ u_1 (x, y) = -\frac{2}{12} y^4 - \frac{2}{3} xy^3 \]  

(2.93)

\[ u_2 (x, y) = -L_y^{-1} \left( L_x (u_1) \right) = -L_y^{-1} \left( L_x \left( \frac{2}{12} y^4 - \frac{2}{3} xy^3 \right) \right) = -L_y^{-1} \left( \frac{2}{3} y^3 \right) \]

\[ \therefore \ u_2 (x, y) = -\frac{2}{12} y^4 \]  

(2.94)
u(x, y) = u_0 + u_1 + u_2 + ... = \frac{2}{3} x y^3 + x^2 y^2 - \frac{2}{12} y^4 - \frac{2}{3} x y^3 + \frac{2}{12} y^4
= x^2 y^2. \quad (2.95)

2.7 \quad u_x + u_y = 2u, u(x,0) = e^x, u(0, y) = e^y, \quad (2.96)

Solution:
In an operator form equation (2.96)

L_y u(x, y) = 2u (x, y) - L_x u (x, y), \quad (2.97)

Applying the inverse operator \( L_y^{-1} \) to both sides of (2.97)

u(x,0) = e^x, u(x, y) = e^x + L_y^{-1} (2u - L_x u), \quad (2.98)

Substituting \( u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \)

\[ \sum_{n=0}^{\infty} u_n(x, y) = e^x + L_y^{-1} \left[ 2 \sum_{n=0}^{\infty} u_n(x, y) - L_x \sum_{n=0}^{\infty} u_n(x, y) \right], \quad (2.99) \]

By considering few terms of the decomposing on of \( u(x, y) \), \( (2.99) \)

\[ u_0 + u_1 + u_2 + ... = e^x + L_y^{-1} \left[ 2(u_0 + u_1 + u_2 + ...) - L_x(u_0 + u_1 + u_2 + ...) \right], \quad (2.100) \]

Proceeding as before we identify the recursive scheme

\[ u_0(x, y) = e^x, \]
\[ u_{k+1}(x, y) = -L_y^{-1} \left( 2(u_{k}) - L_x(u_{k}) \right), \quad (2.101) \]

The components \( u_0, u_1, u_2, ... \)

\[ u_1(x, y) = -L_y^{-1} \left( 2u_0 - L_x u_0 \right) = L_y^{-1} (2e^x - L_x e^x y) \]
\[ = L_y^{-1} (2e^x - e^x y) = e^x y. \quad (2.102) \]

\[ u_2(x, y) = L_y^{-1} (2u_1 - L_x u_1) = L_y^{-1} (2e^x y - L_x e^x y) \]
\[ = e^x y^2 - \frac{e^x y^2}{2} \Rightarrow e^x y^2. \quad (2.103) \]

\[ u_3(x, y) = L_y^{-1} (2u_2 - L_x u_2) = L_y^{-1} \left( 2 \frac{e^x y^2}{2!} - L_x \frac{e^x y^2}{2!} \right) \]
\[ = L_y^{-1} \left( e^x y^2 - \frac{e^x y^2}{2!} \right) = \left( \frac{1}{3} e^x y^3 - \frac{e^x y^3}{3 \times 2} \right) \]
\[ e^y y^3 - \left( \frac{1}{3} - \frac{1}{6} \right) \Rightarrow \frac{e^y y^3}{3!}. \]  

Consequently the solution is given by

\[ u(x, y) = u_0 + u_1 + u_2 + ... = e^x (1 + y + \frac{y^2}{2!} + ...) = e^x y^3 = e^{x+y}. \]  

**2.8**

\[ x u_x + u_y = 2u, u(x, 0) = x, u(0, y) = 0. \]  

**Solution:**

In an operator for equation (2.106)

\[ L_y u(x, y) = 2u(x, y) - xL_x u(x, y), \]  

Applying the inverse operator \( L_y^{-1} \) to both sides of equation (2.107) and using the condition \( u(x, 0) = x \) yields

\[ u(x, y) = x + L_y^{-1} (2u - xL_x u), \]  

Substituting \( u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \) into both sides equation (2.108)

\[ \sum_{n=0}^{\infty} u_n(x, y) = x + L_y^{-1} \left( 2 \sum_{n=0}^{\infty} u_n(x, y) - xL_x \sum_{n=0}^{\infty} u_n(x, y) \right), \]  

By considering few terms of the decomposition of \( u(x, y) \) and equation (2.109) become

\[ u_0 + u_1 + u_2 + ... = x + L_y^{-1} \left( 2(u_0 + u_1 + ...) - xL_x (u_0 + u_1 + ...) \right), \]  

Proceeding as before we identify the recursive equation (2.110) scheme \( u_0(x, y) = x \)

\[ u_{k+1}(x, y) = L_y^{-1} (2 u_k - xL_x u_k), k \geq 0. \]

The components \( u_0, u_1, u_2, ... \) are determined as follower \( u_0(x, y) = x \)

\[ u_1(x, y) = L_y^{-1} (2 u_0 - xL_x u_0) = L_y^{-1} (2x - xL_x x), u_1(x, y) = xy, \]  

\[ u_2(x, y) = L_y^{-1} (2 u_1 - xL_x u_1) = L_y^{-1} (2xy - xL_x xy), u_2(x, y) = \frac{xy^2}{2!}, \]  

\[ u_3(x, y) = L_y^{-1} (2 u_2 - xL_x u_2) = L_y^{-1} \left( \frac{2xy^2}{2!} - xL_x \frac{xy^2}{2!} \right) = \frac{xy^3}{3!}, \]  

Consequently the solution is given by:

\[ u(x, y) = u_0 + u_1 + u_2 + ... = x \left( 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + ... \right) = x e^y. \]
2.9 \[ u_x + u_y + u_z = 3u \]
\[ u(0, y, z) = e^{yz} , \quad u(x, 0, z) = e^{xz} , \quad u(x, y, 0) = e^{xy} \] \hspace{1cm} (2.116)

**Solution:**

**X – Solution:**

Where \( u = u(x, y, z) \)

In an operator form equation (2.116) can be rewritten as

\[ L_x u(x, y, z) = 3u - L_y u - L_z u , \] \hspace{1cm} (2.117)

Where the operator \( L_x , L_y \) and \( L_z \) are defined

\[ L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, L_z = \frac{\partial}{\partial z} , \] \hspace{1cm} (2.118)

Assume that the operator \( L_x \) is invertible and the inverse operator \( L_x^{-1} \) in an indefinite integral from 0 to \( x \) applying the integral \( L_x^{-1} \) to both sides of equation (2.117) and using given condition that \( u(0, y, z) = e^{yz} \) yields

\[ u(x, y, z) = e^{yz} + L_x^{-1}(3u - L_y u - L_z u) , \] \hspace{1cm} (2.119)

Proceeding as before we substituting the decomposition

\[ u(x, y, z) = \sum_{n=0}^{\infty} u_n(x, y, z) , \] \hspace{1cm} (2.120)

into both of equation (2.119) to find

\[ \sum_{n=0}^{\infty} u_n(x, y, z) = e^{yz} + L_x^{-1}\left(3\sum_{n=0}^{\infty} u_n - L_y \sum_{n=0}^{\infty} u_n - L_z \sum_{n=0}^{\infty} u_n \right) , \] \hspace{1cm} (2.121)

Using few terms of the decomposition of \( u(x, y, z) \) equation (2.121) becomes

\[ u_0 + u_1 + u_2 + \ldots = e^{yz} + L_x^{-1}\left(3(u_0 + u_1 + \ldots) - L_y(u_0 + u_1 + \ldots) - L_z(u_0 + u_1 + \ldots) \right) \] \hspace{1cm} (2.122)

The components \( u_0 , u_1 , u_2 , \ldots \) can be determined recently as follows

\[ u_0(x, y, z) = e^{yz} . \] \hspace{1cm} (2.123)

\[ u_1(x, y, z) = L_x^{-1}(3u_0 - L_y u_0 - L_z u_0) \]
\[ = L_x^{-1}(3e^{yz} - L_y e^{yz} - L_z e^{yz}) = L_x^{-1}(3e^{yz} - e^{yz} - e^{yz}) \]
\[ = 3xe^{yz} - xe^{yz} - xe^{yz} = xe^{yz} . \] \hspace{1cm} (2.124)

\[ u_2(x, y, z) = L_x^{-1}(3xe^{yz} - L_y xe^{yz} - L_z xe^{yz}) \]
\[ u(x, y, z) = L_x^{-1} (3xe^{yz} - xe^{yz} - xe^{yz}) = \frac{x^2 e^{yz}}{2!}. \quad (2.125) \]

\[ u_3(x, y, z) = L_x^{-1} \left( \frac{3x^2}{2!} e^{yz} - x^2 L_y \frac{e^{yz}}{2!} - x^2 L_z \frac{e^{yz}}{2!} \right) \]

\[ = L_x^{-1} \left( \frac{3x^2}{2!} e^{yz} - x^2 e^{yz} - x^2 e^{yz} \right) = \frac{x^3 e^{yz}}{3!}. \quad (2.126) \]

\[ \therefore u(x, y, z) = u_0 + u_1 + u_2 = e^{yz} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right) = e^{yz} e^x = e^{x+yz}. \quad (2.127) \]

**Y- Solution:**

\[ u(x, y, z) = e^{yz}, \quad (2.128) \]

\[ L_x u(x, y, z) = 3u - L_x u - L_z u, \]

Where

\[ L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}, \quad L_z = \frac{\partial}{\partial z}. \quad (2.129) \]

\[ \therefore u(x, y, z) = e^{yz}, \quad (2.130) \]

\[ \therefore u(x, y, z) = e^{yz} + L_x^{-1} (3u - L_x u - L_z u), \quad (2.131) \]

\[ \therefore u(x, y, z) = \sum_{n=0}^{\infty} u_n(x, y, z), \quad (2.132) \]

\[ \sum_{n=0}^{\infty} u_n(x, y, z) = e^{yz} + L_y^{-1} \left( 3 \sum_{n=0}^{\infty} u_n - L_x \sum_{n=0}^{\infty} (u_n) - L_z \sum_{n=0}^{\infty} (u_n) \right). \quad (2.133) \]

\[ u_0 + u_1 + u_2 + \ldots = e^{yz} + L_y^{-1} (3(u_0 + u_1 + \ldots) - L_x (u_0 + u_1 + \ldots) - L_z (u_0 + u_1 + \ldots)) \]

\[ u_0 = e^{yz}. \quad (2.134) \]

\[ u_1(x, y, z) = L_x^{-1} \left( 3u_0 - L_x u_0 - L_z u_0 \right) = L_y^{-1} \left( 3e^{yz} - e^{yz} - e^{yz} \right) = ye^{yz}. \quad (2.135) \]

\[ u_2(x, y, z) = L_y^{-1} \left( 3u_1 - L_x u_1 - L_z u_1 \right) = L_x^{-1} \left( 3ye^{yz} - ye^{yz} - ye^{yz} \right) = \frac{y^2}{2!} e^{yz} \quad (2.136) \]

\[ u(x, y, z) = e^{yz} (1 + y + \frac{y^2}{2!} + \ldots) = e^{x+yz}. \quad (2.137) \]
2.10
\[
\begin{align*}
\begin{cases}
  u_x + u_y + u_z = 3 \\
  u(0, y, z) = y + z, u(x, 0, z) = x + z, u(x, y, 0) = x + y
\end{cases}
\end{align*}
\]

\[ (2.138) \]

**Solution:**

**Z–solution:**

Where \( u = u(x, y, z) \)

In an operator form equation (2.138) can be rewritten as

\[
L u(x, y, z) = 3 - L_x u - L_y u,
\]

\[ (2.139) \]

Where the operators \( L_x, L_y \) and \( L_z \) are defined by

\[
L_x = \frac{\partial}{\partial x}, L_y = \frac{\partial}{\partial y}, L_z = \frac{\partial}{\partial z},
\]

\[ (2.140) \]

\( \therefore u(x, y, z) = x + y \Rightarrow u(x, y, z) = x + y + L_z^{-1}(3 - L_x u - L_y u), \)

\[ (2.141) \]

\[
u(x, y, z) = \sum_{n=0}^{\infty} u_n(x, y, z),
\]

\[ (2.142) \]

both sides of equation (2.141) to find

\[
\sum_{n=0}^{\infty} u_n(x, y, z) = x + y + L_z^{-1}\left(3 - L_x \sum_{n=0}^{\infty} (u_n) - L_y \sum_{n=0}^{\infty} (u_n)\right),
\]

\[ (2.143) \]

Using few of the decomposition of \( u(x, y, z) \), equation (2.143)

\[
u_0 + u_1 + u_2 + ... = x + y + L_z^{-1}(3 - L_x (u_0 + u_1 + ...) - L_y (u_0 + u_1 + ...),
\]

\[ (2.144) \]

\[
u_0(x, y, z) = x + y
\]

\[ (2.145) \]

\[
u_{k+1}(x, y, z) = L_z^{-1} (3 - L_x u_k - L_y u_k), k \geq 0,
\]

\[ (2.146) \]

\[
u_1(x, y, z) = L_z^{-1} (3 - L_x u_0 - L_y u_0) = L_z^{-1}(3 - 1 - 1) = z,
\]

\[ (2.147) \]

\[
u_2(x, y, z) = L_z^{-1} (3 - L_x u_1 - L_y u_1) = L_z^{-1}(3) = 3z,
\]

\[ (2.148) \]

\[
u_3(x, y, z) = L_z^{-1}(3) = 3z
\]

\[ (2.149) \]

\[
u(x, y, z) = u_0 + u_1 + u_2 + ... = x + y + z + 3z + ...
\]

\[ (2.150) \]
CHAPTER THREE

THE MODIFIED DECOMPOSITION METHOD

3.1 Technique of the method:

In this section we will introduce a reliable modification of the Adomian developed by Wzwaz and, the modified decomposition method will further accelerate the convergence of the series solution. It is to be noted that the modified decomposition method will be applied. Wherever it is appropriate, to all partial differential equation of any order [12],[1].

The modification will be outlined in this section and will be employed in this section and in other chapters as well. To give clear description of the technique, we consider the partial differential equation in an operator form.

\[ Lu + Ru = g, \quad (3.1) \]

Where \( L \) is the highest order derivative, \( R \) is a linear differential operator of less order or equal order to \( L \), and \( g \) is the source term, operating with the inverse operator \( L^{-1} \) on (3.1) we obtain.

\[ u = f - L^{-1}(Ru), \quad (3.2) \]

Where \( f \) represents the terms arising from the given initial condition and from integrating the source term \( g \), we then proceed as discussed is in section above and define the solution \( u \) as an infinite sum of components defined by

\[ u = \sum_{n=0}^{\infty} u_n, \quad (3.3) \]

The aim of the decomposition method is to determine the components \( u_n, n \geq 0 \) recurrently and elegantly. To achieve this goal, the decomposition method admits

\[ u_0 = f, \quad (3.4) \]

\[ u_{k+1} = -L^{-1}(Ru_k), k \geq 0. \]

In view of equation (3.4), the components \( u_n, n \geq 0 \) are readily obtained. The modified decomposition method introduces a slight variation to the recursive relation equation (3.4) that will lead to the determination of the components of \( u \) in a faster and easier way. For specific cases, the function \( f \) can be set as the sum of two partial functions, namely \( f_1 \) and \( f_2 \). In other words, we can set
\[ f = f_1 + f_2, \quad (3.5) \]

Using equation (3.5) we introduce a qualitative change in the formation of the recursive relation equation (3.4). To reduce the size of calculations, we identify the zeroth component \( u_0 \) by on part of \( f \), namely \( f_1 \) or \( f_2 \). The other part of \( f \) can be added to the components \( u_1 \) among other terms. In other words, the modified recursive relation can be identified by

\[
\begin{align*}
    u_0 &= f_1, \\
    u_1 &= f_2 - L^1(Ru_0), \\
    u_{k+1} &= -L^1(Ru_k), k \geq 1.
\end{align*}
\]

(3.6)

An important point can be made here in that we suggest a change in the formation of the first two components \( u_0 \) and \( u_1 \) only. Although this variation in the formation of \( u_0 \) and \( u_1 \) is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of calculations. Two important remarks related to the modified method can be made here first, by proper selection of the functions, \( f_1 \) and \( f_2 \), the exact solution \( u \) may be obtained by using very few iterations, and sometimes by evaluating only two components, the success of this modification depends only on the choice of \( f_1 \) and \( f_2 \), and this can be made through trials. Second if \( f \) consists of one term only, the standard decomposition method should be employed in this case. It is worth mentioning that the modified decomposition method will be used for linear and nonlinear equations of any order. In the coming chapters, it will be used wherever it is appropriate. The modified decomposition method will be illustrated by discussing the following problems.

3.2 Problems

3.2.1 Use the modified decomposition method to solve the first order partial differential equation:

\[ u_x + u_y = 3x^2y^3 + 3x^3y^2, \quad u(0, y) = 0, \quad (3.7) \]

Solution:

In an operator form equation (3.7) becomes

\[ L_u = 3x^2y^3 + 3x^3y^2 - u_y, \quad (3.8) \]

Where \( L_u \) is first order partial derivative with respect to \( x \). Applying inverse operator \( L_x^{-1} \) to both sides of equation (3.8) gives
\[ u(x, y) = x^3 y^3 + \frac{3}{4} x^4 y^2 - L_x^{-1} (u_y), \quad (3.9) \]

The function \( f(x, y) \) consists of two terms, hence we set
\[
\begin{align*}
  f_1(x, y) & = x^3 y^3, \\
  f_2(x, y) & = \frac{3}{4} x^4 y^2,
\end{align*}
\]
\[ (3.10) \]

In view of equation (3.10) we introduce the modified recursive relation
\[
\begin{align*}
  u_0(x, y) & = x^3 y^3, \\
  u_1(x, y) & = \frac{3}{4} x^4 y^2 - L_x^{-1} (u_0)_y, \\
  u_{k+1}(x, y) & = -L_x^{-1} (u_k)_y, \quad k \geq 1.
\end{align*}
\]
\[ (3.11) \]

This gives
\[
\begin{align*}
  u_0(x, y) & = x^3 y^3, \\
  u_1(x, y) & = \frac{3}{4} x^4 y^2 - L_x^{-1} (3x^3 y^2) = 0, \\
  u_{k+1}(x, y) & = 0, k \geq 1.
\end{align*}
\]
\[ (3.12) \]

It then follows that the solution is
\[ u(x, y) = x^3 y^3, \quad (2.13) \]

This example clearly shows that the solution can be obtained by using two iterations and hence the volume of calculations is reduced.

**3.2.2 Use the modified decomposition method to solve the first order partial differential equation:**

\[ u_x - u_y = x^3 - y^3, u(0, y) = \frac{1}{4} y^4, \quad (3.14) \]

**Solution:**

In an operator form equation (3.14) becomes
\[ L_x u = x^3 - y^3 + u_y, \quad (3.15) \]

Where \( L_x \) is a first order partial derivative with respect to \( x \), proceeding as before we obtain
\[
\begin{align*}
  u(x, y) & = \frac{1}{4} y^4 + \frac{1}{4} x^4 - xy^3 + L_x^{-1} (u_y),
\end{align*}
\]
\[ (3.16) \]

We next split the function \( f(x, y) \) as follows
\[ f_1(x, y) = \frac{1}{4} x^4 + \frac{1}{4} y^4, \quad \text{(3.17)} \]

\[ f_2(x, y) = -xy^3 \]

Consequently, we set the modified recursive relation

\[ u_0(x, y) = \frac{1}{4} x^4 + \frac{1}{4} y^4, \]

\[ u_1(x, y) = -xy^3 + L_x^{-1}(u_0), \quad \text{(3.18)} \]

\[ u_{k+1}(x, y) = L_x^{-1}(u_k), \quad k \geq 1. \]

This gives

\[ u_0(x, y) = \frac{1}{4} x^4 + \frac{1}{4} y^4, \]

\[ u_1(x, y) = -xy^3 + L_x^{-1}(y^3) = 0, \quad \text{(3.19)} \]

\[ u_{k+1}(x, y) = 0, \quad k \geq 1. \]

The exact

\[ u(x, y) = \frac{1}{4} x^4 + \frac{1}{4} y^4. \quad \text{(3.20)} \]

Follows immediately.

**3.2.3 Use the modified decomposition method to solve the first partial differential equation:**

\[ u_x + u_y = u, \quad u(0, y) = 1 + e^y, \quad \text{(3.21)} \]

**Solution:**

Operating with the inverse operator \( L_x^{-1} \) on equation (3.21) and using the given condition gives

\[ u(x, y) = 1 + e^y + L_x^{-1}(u - u_y), \quad \text{(3.22)} \]

We next split function \( f(x, y) \) as follows

\[ f_1(x, y) = e^y, \]

\[ f_2(x, y) = 1, \quad \text{(3.23)} \]

To determine the components \( u(x, y) \), we set the modified recursive relation
\( u_0(x, y) = e^x, \)

\( u_1(x, y) = 1 + L_y^{-1}(u_0 - (u_0)_y), \)

\( u_{k+1}(x, y) = L_y^{-1}(u_k - (u_k)_y), \quad k \geq 1. \)

This gives

\( u_0(x, y) = e^x, \)

\( u_1(x, y) = 1, \quad \text{(3.25)} \)

\( u_2(x, y) = x, \)

\( u_3(x, y) = \frac{x^2}{2!}, \)

And so on the solution in a series form is given by

\[ u(x, y) = e^x + (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...) \quad \text{(3.26)} \]

And in a closed form by

\[ u(x, y) = e^x + e^y. \quad \text{(3.27)} \]

### 3.2.4 Use the modified decomposition method to solve the first order partial differential equation:

\( (u_x + u_y) = \cosh x + \coch y, u(x, 0) = 0, \)

**Solution:**

To effectively use the given condition, we rewrite equation (3.28) in an operator form by

\[ L_y u = \cosh x + \coch y - u_x, \quad \text{(3.29)} \]

Applying the inverse operator \( L_y^{-1} \) on equation (3.29) and using the given condition gives

\[ u(x, y) = \sinh x + \sinh y + y \cosh x - L_y^{-1}(u_x), \quad \text{(3.30)} \]

The function \( f(x, y) \) can be written as \( f_1 + f_2 \) where

\[ f_1(x, y) = \sinh x + \sinh y, \]

\[ f_2(x, y) = y \cosh x, \quad \text{(3.31)} \]

To determine the components of \( u(x, y) \), we set the modified recursive relation
\[ u_0(x, y) = \sinh x + \sinh y, \]
\[ u_1(x, y) = y \cosh x - L_y^{-1}((u_0)_x) = 0, \quad (3.32) \]
\[ u_{k+1}(x, y) = -L_y^{-1}((u_k)_x) = 0, \quad k \geq 1. \]

The exact solution
\[ u(x, y) = \sinh x + \sinh y, \quad (3.33) \]
follows immediately.

It is interesting to point out that two iterations only were used to determine the exact solution, however using the following formation
\[ f_1(x, y) = \sinh x, \]
\[ f_2(x, y) = \sinh y + y \cosh x, \quad (3.34) \]
For \( f(x, y) \) will give the following recursive relation
\[ u_0(x, y) = \sinh x, \]
\[ u_1(x, y) = \sinh y + y \cosh - L_y^{-1}(\cosh x) = \sinh y, \quad (3.35) \]
\[ u_{k+1}(x, y) = 0, \quad k \geq 1. \]

It is obvious from equation (3.35) that all components \( u_j = 0, \quad j \geq 2 \) consequently, the exact solution is
\[ u(x, y) = \sinh x + \sinh y, \quad (3.36) \]

obtained by using the first two components only.

\subsection*{3.2.5}
\[ u_x + u_y = 3x^2 + 3y^2, \quad u(0, y) = y^3, \quad (3.37) \]

\textbf{Solution:}

In an operator form equation (3.37) becomes
\[ L_x u = 3x^2 + 3y^2 \quad u_y, \quad (3.38) \]

Where \( L_x \) a first order partial derivative with respect to \( x \) proceeding as before we obtain
\[ u(x, y) = y^3 + x^3 + 3y^2 x - L_x^{-1}(u_y), \quad (3.39) \]

We next split the function \( f(x, y) \) as follows
\[ f_1(x, y) = y^3 + x^3, \quad (3.40) \]
\[ f_2(x, y) = 3y^2x, \]

Consequently we set the modified recursive relation
\[
\begin{align*}
    u_0(x, y) &= y^3 + x^3, \\
    u_1(x, y) &= 3y^2x - L_x^{-1}(u_y), \\
    u_{k+1}(x, y) &= L_x^{-1}(u_y), k \geq 1.
\end{align*}
\]

This gives
\[
\begin{align*}
    u_0(x, y) &= y^3 + x^3, \\
    u_1(x, y) &= 3y^2x - L_x^{-1}(3y^2) = 3y^2x; 3y^2x = 0, \\
    u_{k+1}(x, y) &= 0, k \geq 1.
\end{align*}
\]

The exact solution
\[
    u(x, y) = y^3 + x^3.
\]

followers immediately.

3.2.6
\[
    u_x + u_y = u, \quad u(0, y) = 1 - e^y, \quad (3.44)
\]

Solution:

Operator with the inverse operator \( L_x^{-1} \) on (3.44) and using the given condition gives
\[
    u(x, y) = 1 - e^y + L_x^{-1}(u - u_y), \quad (3.45)
\]

We next split function \( f(x, y) \) as follows
\[
\begin{align*}
    f_1(x, y) &= -e^y, \\
    f_2(x, y) &= 1.
\end{align*}
\]

To determine the components of \( u(x, y) \) we set modified recursive relation
\[
\begin{align*}
    u_0(x, y) &= -e^y, \\
    u_1(x, y) &= 1 + L_x^{-1}(u_0 - (u_0)_y), \\
    u_{k+1}(x, y) &= L_x^{-1}(u_k - (u_k)_y), k \geq 1.
\end{align*}
\]

This gives
\[
\begin{align*}
    u_0(x, y) &= -e^y, \\
    u_1(x, y) &= 1, \\
    u_2(x, y) &= x.
\end{align*}
\]
\[ u_0(x, y) = \frac{x^2}{2!}, \]

And so on the solution in a series form is given by
\[ u(x, y) = -e^y + (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) \quad (3.49) \]

And closed form by
\[ u(x, y) = e^x - e^y. \quad (3.50) \]

### 3.2.7

\[ u_x - yu = 0, u(0, y) = 1. \quad (3.51) \]

**Solution:**

Operating with the inverse operator \( L_x^{-1} \) on equation (3.51) and using the given condition gives
\[ u(x, y) = 1 + L_x^{-1}(yu), \quad (3.52) \]

We next split function \( f(x, y) \) as follows
\[ f_1(x, y) = 1, \quad (3.53) \]
\[ f_2(x, y) = 0, \]

To determine components of \( u(x, y) \), we set the modified recursive relation
\[ u_0(x, y) = 1, \]
\[ u_1(x, y) = L_x^{-1}y(u_0), \quad (3.54) \]
\[ u_{k+1}(x, y) = L_x^{-1}y(u_k), \quad k \geq 1. \]

This gives
\[ u_0(x, y) = 1, \]
\[ u_1(x, y) = xy, \]
\[ u_2(x, y) = L_x^{-1}y(xy) = \frac{x^2}{2!}y^2, \quad (3.55) \]
\[ u_3(x, y) = L_x^{-1}y\left(\frac{x^2y^2}{2!}\right) = \frac{x^3}{3!}y^3. \]

And so on, the solution is a series form is given by
\[ u(x, y) = 1 + xy + \frac{x^2y^2}{2!} + \frac{x^3y^3}{3!} + \ldots. \quad (3.56) \]
CHAPTER FOUR

ADOMIAN DECOMPOSITION METHOD FOR SOLUTION of PARABOLIC EQUATION TO NONLOCAL CONDITION

4.1 Introduction

The Adomian decomposition method has been applied to a wide range of problems in physics, chemical reactions and biology. The method provides the solution in rapid convergent series with computable terms. The method was successfully applied to nonlinear delay equations nonlinear dynamic systems, the wave equation and the heat equation, coupled nonlinear partial differential equation linear and nonlinear integro-differential equation and Airy's equation different modifications of this method and their applications are given there.

We consider the following parabolic equation

\[
\frac{\partial}{\partial t} \left( u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} \right) - \frac{\partial^3 u(x,t)}{\partial x^3} = f(x,t), \quad x \in \left[0,1\right], \quad t > 0
\]  

(4.1)

Subject to the initial condition

\[ u(x,0) = \psi(x), \quad x \in \left[0,1\right], \]  

(4.2)

And the nonlocal conditions

\[
\begin{cases}
  u(x,0) = \int_0^1 g(x) u(x,t) \, dx + r(t) \\
  u(x,0) = \int_0^1 h(x) u(x,t) \, dx + v(t)
\end{cases}
\]

(4.3)

for \( t > 0 \)

Where \( x \) and \( t \) are the spatial and time coordinates respectively, \( u \) is the unknown function of \( x \) and \( t \) to be determined and \( f, \psi, g, h, r, \) and \( v \) are suitably prescribed. S.A Duby study the same problem with the laplace transform method .we propose a new technique easily, quickly, and elegantly for obtain analytic solution [16],[11].

4.2 Analysis of the method (ADM):

Let us rewrite the parabolic equation (4.1) in the standard operator form:

\[
L_y (u - L_{xy}(u)) - L_{xxy}(u) = f(x,t),
\]  

(4.4)
Where \( L_\gamma(.) = \frac{\partial}{\partial t}(.), \ L_{xx}(.) = \frac{\partial^2}{\partial x^2}(.) \), the inverse operator of the operator \( L_\gamma \) exists and it is defined as

\[
L_\gamma^{-1}(.) = \int_0^t(.)dt. \tag{4.5}
\]

Thus, applying the inverse operator \( L_\gamma^{-1} \) on both sides of (4.4) and using the initial condition yields

\[
L_\gamma^{-1} L_\gamma(u) = L_\gamma^{-1} L_{\alpha t}(u) + L_\gamma^{-1} L_{xx}(u) + L_\gamma^{-1} f(x,t), \tag{4.6}
\]

Therefore, it follows that

\[
u(x,t) = \psi(x) + L_{\alpha t}\left(u(x,t)\right) - L_{xx}\left(u(x,t)\right)\bigg|_{t=0} + L_\gamma^{-1} L_{\alpha t}(u(x,t)) + L_\gamma^{-1} f(x,t). \tag{4.7}\]

Now, we decompose the unknown function \( u(x,t) \) a sum of components defined by the series:

\[
u(x,t) = \sum_{n=0}^{\infty}u_n(x,t), \tag{4.8}\]

The zeros components is usually taken to be all terms arise from the initial condition and the integration of the source term \( f(x,t) \), i.e.,

\[
u_0(x,t) = \psi(x) + L_\gamma^{-1} f(x,t) \tag{4.9}\]

The remaining components \( u_n(x,t) = \psi, n \geq 1 \), can be completely determined such that each term is computes by using the previous term. Since \( u_0 \) is known,

\[
u_n(x,t) = L_\gamma^{-1} L_{xx}\left(u_{n-1}(x,t)\right) + L_{xx}\left(u_{n-1}(x,t)\right) - L_{xx}\left(u_{n-1}(x,t)\right)\bigg|_{t=0}, \ n \geq 1. \tag{4.10}\]

Finally, an N-term approximate solution is given by

\[
\phi_N(x,t) = \sum_{n=0}^{N-1}u_n(x,t), \ N \geq 1. \tag{4.11}\]

And the exact solution is \( u(x,t) = \lim_{n \to \infty} \phi_N \).

### 4.3 Examples

In this chapter, we apply the Adomain decomposition method in following examples
Example (4.1): we consider the parabolic problem (4.1), (4.3) with
\[
    f(x,t) = \left( x(x-1) - \frac{1}{7} \right) \exp(-t), \psi(x) = x(1-x) + \frac{1}{7}
\]
\[
g(x) = h(x) = \frac{6}{13}, r(t) = v(t) = 0.
\]
We can rewrite recursive formula as
\[
\begin{align*}
    u_0(x,t) &= \psi(x) + L_t^{-1} f(x,t) \\
    u_n(x,t) &= L_t^{-1} L_x u_{n-1}(x,t) + L_x u_n(x,t) - L_x u_{n-1}(x,t) \bigg|_{t=0}, n \geq 1
\end{align*}
\]
which gives
\[
    u_0(x,t) = \left( x(1-x) + \frac{1}{7} \right) \exp(-t)
\]
\[
u_n(x,t) = 0, n \geq 1
\]
The solution in the series form is given by
\[
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)
\]
Or
\[
u(x,t) = u_0(x,t) + u_n(x,t), n \geq 1
\]
Which gives the solution of (4.1), (4.3) with (4.12) as
\[
u(x,t) = \left( x(1-x) + \frac{1}{7} \right) \exp(-t).
\]
Example (4.2): we consider the parabolic problem (4.1), (4.3) with
\[
f(x,t) = -\frac{2(x^2 + t + 1)}{(t+3)^3}, \psi(x) = \frac{x^2}{9},
\]
\[
g(x) = h(x) = x, r(t) = -\frac{1}{4(t+3)^2}, v(t) = \frac{3}{4(t+3)^2}.
\]
we can rewrite the recursive formula is
\[
\begin{align*}
    u_0(x,t) &= \psi(x) + L_t^{-1} f(x,t) \\
    u_n(x,t) &= L_t^{-1} L_x u_{n-1}(x,t) + L_x u_n(x,t) - L_x u_{n-1}(x,t) \bigg|_{t=0}, n \geq 1
\end{align*}
\]
Which gives
\[
u_0(x,t) = \frac{x^2}{9} + \int_0^t -\frac{2(x^2 + t + 1)}{(t+3)^3} \, dt
\]
\[
\begin{align*}
&= \frac{x^2}{9} - 2 \int_0^t \left( \frac{(t+3)}{(t+3)^3} + \frac{(x^2 - 2)}{(t+3)^3} \right) \, dt \\
&= \frac{x^2}{9} - 2 \int_0^t \left( (t+3)^{-2} + (x^2 - 2)(t+3)^{-3} \right) \, dt \\
&= \frac{x^2}{9} - 2 \left[ \frac{-1}{t+3} - \frac{2(x-2)}{(t+3)^2} \right] \bigg|_0^t \\
\therefore \ u_0(x,t) &= \frac{x^2 + 2t + 4}{(t+3)^2} - \frac{4}{9} \quad (4.21) \\
\end{align*}
\]

\[
\begin{align*}
&= L^L_{x} \left( \frac{x^2 + 2t + 4}{(t+3)^2} - \frac{4}{9} \right) + L_{x} \left( \frac{x^2 + 2t + 4}{(t+3)^2} - \frac{4}{9} \right) - L_{x} \left( \frac{x^2 + 2t + 4}{(t+3)^2} - \frac{4}{9} \right) \bigg|_{t=0} \\
\therefore \ u_1(x,t) &= -\frac{2t - 4}{(t+3)^2} + \frac{4}{9} \quad (4.22) \\
\end{align*}
\]

\[
\begin{align*}
u_n(x,t) &= 0 \quad n \geq 2 \\ (4.23)
\end{align*}
\]

The solution in the series forms is given by

\[
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad , \quad (4.24)
\]

Or

\[
u(x,t) = u_0(x,t) + u_1(x,t) + u_n(x,t), \quad n \geq 2 \\ (4.25)
\]

Hence the solution of equations (4.1),(4.3) with equation (4.19) is given as

\[
u(x,t) = \frac{x^2}{(t+3)^2} \quad . \quad (4.26)
\]
4.4 Conclusion and Recommendations:

In this research we will discuss the first order linear partial differential equation, homogeneous and nonhomogeneous, partial differential equations of first order are used to model traffic flow on a crowded road, blood flow through an elastic-walled tide shock waves, and special cases of the general theories of gas dynamics and hydraulics.

A comparative study between the method of characteristics and the other two methods will be carried out through illustrative examples. In this study we show that application of Adomian decomposition method (ADM) to a class of partial differential equations, this method provide us a straightforward, accurate and quite efficient technique in comparison with the other usual classical methods. The result are also verified on two examples discussed at the end of the study.

Finally; We recommend to apply the Adomian decomposition method for solving system of partial differential equations of higher orders.
References: