Adomian Decomposition Method for Solving First and Second Order linear Partial Differential Equations

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A Dissertation
Submitted to the University of Gezira in partial Fulfillment of the Requirements for the Award of the Degree of Master of Science

in

Mathematic Science

Department Mathematic

Faculty of Mathematical and Computer Sciences
Adomian Decomposition method for Solving First and Second Order linear Partial Differential Equations

Mohammad Siddig Ibrahim Ali

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Date: May 2016
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Declaration

I assure you and every body that I am very glad to attain this degree of science for what I exerted my efforts. Also I desire to further my education.
Dedication

Before any thing else, I confess with the help of lecturers which they performed to me. I can reward them with my thanks only, and hope them more bright future. I offer them all my respect and appreciations.
Acknowledgement

I indebted to any person who participated in my study and gave me the chance to attain this degree of education. With my thanks to all these who helped me
Adomian Decomposition Method for Solving first and second Order linear Partial Differential Equations

Mohammad Siddig Ibrahim Ali
M.S.c in Mathematics(7/5/2016)

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Abstract

We will discuss the first and second order liner partial differential equation homogenous and inhomogeneous partial differential equations. It is the concern of this thesis to introduce the recently developed methods to handle partial differential equations in an accessible manner. In this thesis we will apply the Adomian decomposition method. [1-7] and the related phenomenon of the noise terms. [14-17] That will accelerate the rapid-convergency of the solution. The decomposition method and the improvements made by the noise terms phenomenon this method provide the solution in an infinite series form. The obtained series may be converge to a closed form solution. We will present useful tool that will accelerate the convergence of the Adomian decomposition method the noise term phenomenon provides a major advantages in that it demonstrates a fast convergence of the solution it is important to note there that noise terms phenomenon that will be introduced in this section at appear only for inhomogeneous PDES in addition, this phenomenon is applicable to all inhomogeneous PDES of in order and will be used where appropriate in the conin. equation with variable coefficient. The A domino decomposition method is implemented also to solve non- homogenous partial differential equations with a variable coefficient.
طريقة أedomian لحل المعادلات التفاضلية الجزئية الخطية من الرتبة الأولى والثانية

محمد صديق إبراهيم علي

ماجستير العلوم في الرياضيات (7/5/2016م)

قسم الرياضيات
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ملخص الدراسة

سوف نقوم بمناقشة حل المعادلة التفاضلية الخطية الجزئية والمعادلة التفاضلية المتجانسة وغير المتجانسة من الرتبة الأولى والثانية. تشمل هذه المعادلات أسباب الحركة في الطرق المزدوجة، مثل تبديل حركة واسعات الدم في الأوعية الم맥ية، وحالة خاصة بالنظريات العامة بنظرية ديناميكية الغازات وهي مختصة بهذا الموضوع. وذلك لإظهار النظريات المتطورة المكتشفة حديثاً ليتسهل التعامل مع المعادلة التفاضلية الجزئية بطريقة سهلة المنال. هذا سوف نقوم به في حل معادلة سريعة ومباشرة بنظرية أedomian التطورية، الذي يركز على نظرية العناصر المختارة. وهذا سوف يؤدي إلى الوصول للحل بطريقة متطورة وواجابة وهي متطورة لحل المعادلات المستقلة. يمكن أن تحل بالحل المعلن سوف نقوم النماذج لتسهل تسهيل حل نظرية أdomian. تقدم نظرية العناصر المختارة ميزة إضافية لقياسها يتسهل الحل السريع. من المهم ملاحظة نظرية العناصر المختارة التي تقوم بتقديمها في هذا الفصل. تظهر ميزة العناصر المختارة في المعادلات التفاضلية الجزئية غير المتجانسة، إضافة لذلك يسهل تطبيقها في كل المعادلات. تحتوي المعادلات التفاضلية الجزئية المتجانسة في أي رتب. سوف يشرح بطرق مفصلة لاحقاً. تعتبر الصورة العامة للمعادلات التفاضلية الجزئية الأحادية غير المتجانسة ذات المعاملات المختلفة، رئيس أdomian الذي طبقت لحل المعادلة التفاضلية الجزئية غير المتجانسة ذات المعاملات المختلفة ومقارنتها بالحل الصحيح.
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Chapter One
Adomain Decomposition Method

1-1 Introduction:

In this chapter we will discuss the first order linear partial differential equations homogeneous and in homogeneous.

Partial differential equations of first order are used to model traffic flow on a crowded road, blood flow through an elastic-walled tube, shock waves and as special cases of the general theories of gas dynamic and hydraulics. It is concern of this thesis to introduce the recently developed methods to handle–partial differential equations in an accessible manner. In this thesis we will apply the Adomian decomposition method [1-4] and the related phenomenon of the noise terms [14-17] that will accelerate the rapid-Convergence of the solution. The decomposition method and the improvements made by the noise terms phenomenon.

1.2 Adomian Decomposition Method (18):

In this section we will discuss the Adomian decomposition method. The Adomian decomposition method has been receiving much attention in recent years in applied mathematics in general and in the area of serious solutions in particular. The method proved to be powerful effective, and can easily a wide of linear or non linear, ordinary or partial differential equation, and linear and non linear integrates fast convergence of the solution and therefore provides several significant advantages in this text, the method will be successfully used to handle some types of linear partial differential equations that appear in several physical models and scientific applications. The method attacks the problem in a direct way and in a straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion. The Adomian decomposition method was introduced and developed by Gorge Adomian in [1-2] and is well addressed in the literature. A considerable amount of research work has been invested recently in applying this method to wide class of linear and non linear ordinary differential equations, partial differential equations an integrate equations as well. In the non linear case for ordinary differential equations and partial differential equations, the method has the advantage of dealing directly with the problem [3-10] these equations are solved without transforming them to more simple ones. The method avoids linearization, perturbation, discretization or any unrealistic assumptions.
The Adomian decomposition method consists of decomposition method the unknown function \( u(x,y) \) of any equation into a sum of an infinite number of components defined by the decomposition series.

\[
    u(x,y) = \sum_{n=0}^{\infty} u_n(x, y), \quad (1-1)
\]

Where the components \( u_n(x, y), \ n \geq 0 \) are to be determined in a recursive manner the decomposition method concerns itself with finding the components \( u_0, u_1, u_2, \ldots \), individually as well as seen through the text, the determination of these components can be achieved in an easy way through a recursivelation that usually involve simple integrals.

To give a clear overview of Adomian decomposition method, we first consider the linear differential equation written in operator form by

\[
    Lu + Ru = g \quad (1-2)
\]

Where \( L \) is mostly, the lower order derivative which is assumed to be invertible, \( R \), is other linear differential operator, and \( g \) is source term. It is to be noted that the nonlinear differential equations will be presented in chapter 8. We next apply the inverse operator \( L^{-1} \) to both sides of equation (1-2) and using the given condition to obtain.

\[
    u = f - L^{-1}(Rn) \quad (1-3)
\]

Where the function \( f \) represents the terms arising from integrating the source term \( g \) and from using the given conditions that are assumed to be prescribed. As indicated before, Adomian decomposition method define the solutions \( u \) by an infinite series of components given by

\[
    u = \sum_{n=0}^{\infty} u_n \quad (1-4)
\]

where the components \( u_0, u_1, u_2, \ldots \) are usually recurrently determined-substituting (1-4) into both sides of (1-3) leads to

\[
    \sum_{n=0}^{\infty} u_n = f - L^{-1}(R) \quad (1-5)
\]

For simplicity, equation (1-5) can be rewritten as

\[
    u_0 + u_1 + u_2 + u_3 + \ldots = f - L^{-1} \left( R(u_0 + u_1 + u_2 + \ldots) \right) \quad (1-6)
\]

To construct the recursive relation needed for the determination of the components \( u_0, u_1, u_2, \ldots \), it is important to note that Adomian method suggests that the zeroth component \( u_0 \) is usually defined by the function \( f \) described above i-e- by all terms that are not included under the
inverse operator $L^{-1}$, which arises from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by

$$u_0 = f$$

or equivalently

$$u_{k+1} = - L^{-1} (R(u_k)), \quad k \geq 0, \quad (1-7)$$

$$u_0 = f,$$

$$u_1 = - L^{-1} (R(u_0)),$$

$$u_2 = - L^{-1} (R(u_0)), \quad (1-8)$$

$$u_3 = - L^{-1} (R(u_0)),$$

It is clearly seen that the relation (1-8) reduced the differential equation consideration into an elegant determination of compatible components. Having determined these components, we then substitute it into (1.4) to obtain the solution in a series form. It was formally shown by many researchers that if an exact solution exist for the problem then the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was the roughly investigated by many researchers to confirm the rapid convergence of the resulting series – cherru – ault examined the convergence of Adomian method in addition, cherr vault and Adomian presented a new proof of convergence of the method for more details about the proofs presented to discuss the rapid convergence the reader is advised to see the references mentioned above and the references therein. However, for concrete problems, where a closed form solution is not obtainable a truncated number of terms is usually used for numerical purposes. It was also shown by many that the series obtained by evaluating few term gives an approximation of high degree of accuracy if compared with other numerical techniques. It seems reasonable to give a brief outline about the works conducted by Adomian and other researchers in applying Adomians method. Adomian in [1-2] and in many other works introduced his method and applied it to many deterministic and stochastic problems.

Adomian method has attracted to considerable amount of research work. A comparison between the decomposition method and the perturbation technique showed the efficiency of the decomposition method to the tedious work required by the perturbation method, the advantage of the decomposition method over Picard’s method has been emphasized in many works. Also, a comparative study between Adomian method and Taylor series method has been examined to show that the composition method requires less computational work if compared with Taylor serious. Other comparisons with traditional methods such as finite difference method have been
conducted in the literature. It is to be noted that many studies have shown that few terms of decomposition series provide a numerical result of a high degree of accuracy. Reach-employed Adomian’s method to solve differential equations with singular co-efficient such as Legendre’s equation, Bessel’s equations and Hermit’s equation. Moreover, a suitable definition of the operator was used to overcome the difficulty of singular points of Lane Emden equation, many other studies implement the decomposition for differential equations, ordinary, and partial and for integrate equations, linear and non linear, it is normal in differential equations that we seek a closed form solution or a series solution with a proper number of terms.

**Example (1):**

Consider the equation:

\[ u' (x) = u (x) + 1, u (0) = A \]  

\[ u (x) + 1 \]  

In an form Equation \[ (1-9) \]

\[ Lu = u + 1 \]  

\[ (1-10) \]

Where the differential of error or L is given by

\[ L = \frac{d}{dx} \]  

\[ (1-11) \]

\[ L^{-1} (-) = \int_{0}^{x} (-) dx \]  

\[ (1-12) \]

Applying \( L^{-1} \) to both sides \((1-10)\) and using the initial condition we obtain

\[ L^{-1} (Lu) = L^{-1} (u) + 1 \]  

\[ (1-13) \]

So that

\[ L^{-1} = \int_{0}^{x} u (x) \, dx \]  

\[ (1-14) \]

Or equivalently

\[ u(x) = A + 1 + L^{-1} (u) + 1 \]  

\[ (1-15) \]

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) = A - 1 + L^{-1} \sum_{n=0}^{\infty} u_n(x) + 1 \]
\[
\sum_{n=0}^{\infty} u_n(x) = A - 1 + L^{-1}\left(\sum_{n=0}^{\infty} u_n(x) + 1\right)
\]

\[u_0(x) = A - 1\]

\[u_1(x) = L^{-1}(u_0) + 1\]

\[L^{-1} = \int dx = \frac{z^2}{2} A - 1 + 1\]

\[u_1(x) = u_0 + u_1 + u_2 + \ldots\]

In to both sides of (1-15) gives:

in view of (1-16), the following recursive relation follows immediately – consequently, we obtain.

\[u_2(x) = L^{-1}(u(x)) = A \left[\frac{x^2}{2} \right] - 1(1-18)\]

\[u(x) = u_0 + u_1 + u_2 + \ldots\]

an in a closed form by

\[u(x) = (A - 1) e^x + 1 \quad (1-20)\]

**Example (2):**

Solve the equation:

\[u''(x) = xu(x)\]

\[u(0) = A\]

\[u(0) = B\]
solution:

\[ Lu = xu, \quad L = \frac{d^2}{dx^2} \]

\[ L^{-1}(-) = \int_0^x \int_0^x (-) \, dx \, dx \]

\[ L^{-1}(lu) = L^{-1}(xu) \]

\[ L^{-1} = \int_0^x u''(x) \]

\[ = \int_0^x \left( \int_0^x u''(x) \right) \, dx \]

\[ = \int_0^x u'(x) - u'(0) \, dx \]

\[ = [u(x)]_0^x - \int_0^x u''(x) \, dz \]

\[ u(x) - xu'(0) - u(0) \]

\[ \sum_{n=0}^\infty u(x) - xB + A + L^{-1}(xu) \]

\[ u_n(x) = A + xB + L^{-1}(x) \sum_{n=0}^\infty (u_n(x)) \]

\[ u_0(x) = A + xB \]

\[ ur + l(x) = L^{-1}(xu_0) \]

\[ L^{+1} = \int_0^x \, dx \, dx \]

\[ = \int_0^x \int_0^x \frac{x (A + Bx)}{2} \, dx \, dx \]

\[ = \left[ \frac{Ax^2}{2} + \frac{Bx^3}{3} \right]_0^x \]

\[ = \left[ A \frac{x^2}{2} + \frac{Bx^3}{3} \right]_0^x \]

\[ = 15x^3 \]
\[ u_1(x) = A \frac{x^4}{6} + \frac{x^4}{12} \]
\[ u_2(x) = \mathcal{L}^{-1}(xu_1(x)) \]
\[ = \int_{0}^{x} x \left( \frac{x^3}{6} + B \frac{x^4}{12} \right) dx \]
\[ = A \frac{x^5}{6} + B \frac{x^6}{12} \left[ \frac{x^5}{5} \right]_{0}^{x} \]
\[ = A \frac{x^5}{30} + B \frac{x^6}{72} \]
\[ u_2(x) = A \frac{x^6}{180} + B \frac{x^7}{504} \]
\[ u_3(x) = \mathcal{L}^{-1}(xu_2(x)) \]
\[ L^{-1} = \int_{0}^{x} x \left( A \frac{x^6}{180} + B \frac{x^7}{504} \right) dx \]
\[ = A \frac{x^7}{180} + B \frac{x^8}{540} \left[ \frac{x^7}{7} \right]_{0}^{x} \]
\[ = A \frac{x^7}{1440} + B \frac{x^8}{4536} \left[ \frac{x^8}{8} \right]_{0}^{x} \]
\[ = A \frac{x^9}{12960} + B \frac{x^{10}}{45360} \]
\[ u_3(x) = A \frac{x^9}{12960} + B \frac{x^{10}}{45360} \]
\[ u(x) = u_0 + u_1 + u_2 + u_3 + \ldots \]
\[ u(x) = A \left( 1 + \frac{x^4}{6} + \frac{x^6}{180} + \frac{x^9}{12960} \right) + \ldots + B \left( x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} \right) \]

**Example (3):**
solve the following PDE:
\[ u' + 2tu = 2t \exp(-t^2) \]

\[ u(0) = 0 \]

**solution:**

\[ Lu = u \]

\[ L = \frac{d}{dt} \]

\[ L^{-1}t = \int_0^x dt \]

\[ L (u' + 2tu) = 2t \exp(-t^2) \]

\[ L^{-1}L(u' + 2tu) = L^{-1}t(2t \exp(-t^2))u \]

\[ u(t) - (0) = L^{-1}t(2t \exp(-t^2)) \]

\[ u(t) = \sum_{n=0}^{\infty} u_n(t) \]

\[ u(t) \sum_{n=0}^{\infty} u_n(t) = 0 + L^{-1}(2t \exp(-t^2)) \sum_{n=0}^{\infty} u_n(t) \]

\[ u_0(t) = 0 \]

\[ u_1(t) = L^{-1}(2t \exp(-t^2))u_0 \]

\[ = t^2 \exp(-t^2) \]

\[ u(t) = u_0 + u_1 + ... \]

\[ u(t) = t^2 e^{-t^2} \]

(1.3) Adomain decomposition method for the solving of first order partial differential equation.
**Example (4):**

Solve the equation:

\[ u_x + u_y = F(x, y) \]

\[ u(0, y) = g(y) \]

\[ u(x, 0) = h(x) \]

**Solution:**

\[ L_x u + lyu = F(x, y) \]

\[ L_x = \frac{\partial}{\partial x}, Ly \]

\[ L^{-1} x(\cdot) = \int_0^x (-) \, dx \]

\[ L^{-1} y(\cdot) = \int (\cdot) \, dy \]

\[ L^{-1} x L u(x, y) = u(x, y) - (0, y) \]

\[ L^{-1} x L u = L^{-1} x f(x, y) - L^{-1} x lyu \]

\[ u(x, y) = g(y) + L^{-1} x f(x, y) - L^{-1} x lyu \]

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \]

\[ = \sum_{n=0}^{\infty} u_n(x, 4) = g(y) + L^{-1} x (f(x, y)) - L^{-1} x ly(\sum_{n=0}^{\infty} u_n(x, y)) \]

This can be rewritten as:

\[ u_0 + u_1 + u_2 + \ldots = g(y) + L^{-1} x f(x, 4) - L^{-1} x ly (u_0 + u_1 + u_2 + \ldots) \]

\[ u_0(x, y) = g(y) + L^{-1} x f(xy) \]

\[ u k_1(x, y) = - L x^{-1} Ly(u k), k_{710} \]

\[ u_0(x, y) = g(y) + L^{-1} x f(x, y) \]
\[ u_{k+1}(x, y) = -L^{-1}x ly(uk), \quad k_{710} \]

That forms the basis for a complete determination of the components \( u_0, u_1, u_2, \ldots \)
Therefore the components can be easily obtained by:

\[ u_0(x, y) = g(y) + Lx^{-1}(x, y) \]

\[ u_1(x, y) = -Lx^{-1} lyu_0(x, y) \]

\[ u_2(x, y) = -Lx^{-1}(lyu(x, y)) \]

\[ u_3(x, y) = -Lx^{-1}(lyu_2(x, y)) \]

and so on. Thus the components \( U_n \) can be determined recursively as far as we like. It is clear
that the accuracy of the approximation can be significantly improved by simply determining
more components having established the components of \( u(x, y) \), the solution in a series form
follows immediately. However the expression.

\[ \emptyset n = \sum_{r=0}^{n-1} ur(x, y) \quad (1-21) \]

is considered then-term approximation to \( u \) for concrete problems where exact solution, we
usually use the truncated series (1-21) for numerical purpose as indicated earlier the convergence
of Adomian decomposition method has been established by many researchers, but will not be
discussed in this text it is important to note that the solution can be also obtained by finding the \( y \)
solution by applying the inverse operator \( Ly^{-1} \) to both sides of the equation:

\[ Ly = f(x, y) - Lxu \quad (1-22) \]

The quality of the \( x \)- solution and the \( y \)-solution is formally justified and will be examined
through the coming examples.

**Example (5):**

Solve the equation:

\[ u_x + u_y = 2x + 2y \]

\[ u(x, 0) = x2 \]

\[ u(0, y) = y^2 \]

**solution:**

\[ \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \]
\[ Lx = \frac{\partial}{\partial X}, \quad ly = \frac{\partial}{\partial y} \]
\[ L^{-1}x^{(\cdot)} = \int_0^x (-) \, dx \]
\[ L^{-1}x(Lxu) = L^{-1} (2x + 2y) + L^{-1} lyu \]
\[ u(x, y) = u(x, o) - L^{-1}x(2x + 2y) (lyu) \]
\[ u(x, y) = \sum_{n=0}^{\infty} u_n (x, y) \]
\[ = \sum_{n=0}^{\infty} u_n (x, y) = x^2 - L^{-1}x (2x + 2y) ly \sum_{n=0}^{\infty} u_n \]
\[ u_0 (x, y) = x^2 \]
\[ u_n (x, y) = - L^{-1}x (2x + 2y) (lyu_k), \quad k \geq 0 \]
\[ u_1 (x, y) = - L^{-1}x(2x + 2y) (lyu_0) \]
\[ - L^{-1}x = \int_0^x 2x + 2y \, dx \]
\[ = x^2 + 2xy \]
\[ u_1 (x, y) = - x^2 + 2xy \]
\[ u_2(x, y) = - L^{-1}x (2x + 2y)(lyu_1), \quad k \geq 0 \]
\[ u_2(x, y) = o \]
\[ u(x, y) = u_0 (x, y) + y(x, y) + u_2(x, y) + ... \]
\[ = x^2 - x^2 + 2xy + 0 \]
\[ u(x, y) = 2xy \]
\[ (2) \ L_yu(x, y) = (2x + 2y) - Lxu(x, y) \]
\[ L_y = \frac{\partial}{\partial y} \]
\[ L^{-1} \gamma (lyu(x, y)) = L^{-1} y (2x + 2y) - L^{-1} y (lxu) \]

\[ u(x, y) = u(0, y) - L^{-1} (2x + 2y) (lxu) \]

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \]

\[ = \sum_{n=0}^{\infty} U_n(x, y) = y^2 L^{-1} y (2x + 2y) \sum_{n=0}^{\infty} u_n \]

\[ u_0(x, y) = y^2 \]

\[ u_k(x, y) = - L^{-1} y (2x + 2y) (lxu_0) \]

\[ -L^{-1} = \int_{0}^{x} = (2x + 2y) = 2xy - y^2 \]

\[ u_1(x, y) = 2xy - y^2 \]

\[ u_2(x, y) = - L^{-1} y (2x + 2y) (lxu) K_{\geq 0} \]

\[ y(x, y) = o \]

\[ u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) + ... = y^2 + 2xy - y^2 + 0 \]

\[ u(x, y) = 2xy. \]
Example (6):

Solve the equation.

\[ u_x + y_u = 0 \]

\[ u(x, 0) = 1 \]

\[ u(0, y) = 1 \]

**solution:**

\[ Lxu(x, y) = 4u(x, y) \]

\[ Lx = \frac{\partial}{\partial x} \]

\[ L^{-1}x Lxu(x, y) = L^{-1}x yu(x, y) \]

\[ u(x, y) = 1 + L^{-1}x yu(x, y) \]

\[ u(x, y) = \int_0^x u_n(x, y) \]

\[ = \int_0^x u_n(x, y) = 1 + L^{-1}x(4)(\int_0^x u_n(x, y) \]

\[ u_{0}(x, y) = 1 \]

\[ u_{n+1}(x, y) = L^{-1}x(yu_{k}) \quad k \geq 0 \]

\[ u_1(x, y) = L^{-1}x(yu_0) \]

\[ L^{-1}x = \int_0^x ydx = yx \]

\[ u_1(x, y) = yx \]

\[ u_2(x, y) = L^{-1}x yu_1 \]

\[ L^{-1}x = \int_0^x yxdx = \frac{1}{2} y^2 x^2 \]

\[ u_3(x, y) = L^{-1}xyu_2 \]

1
Example (7):

Solve the equation.

\[ u_x + u_y = u \]
\[ u(x, 0) = 1 + e^x \]
\[ u(0, y) = 1 + e^y \]

solution:

\[ L_x u(x, y) = u - lyu \]
\[ L_x = \frac{\partial}{\partial x} \]
\[ L_x \int u(x, y) = L_x (u - lyu) \]
\[ u(x, y) = 1 + e^x + L_x (u - lyu) \]

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \]

\[ n=0 \]

\[ = \sum_{n=0}^{\infty} n u_n(x, y) = 1 + e^x + L_x (u - lyu) \]

\[ u_0(x, y) = 1 + e^x \]
\[ u_{k+1} (x, y) = L_x (u_k - lyu_k), k > 0 \]
\[ u_1 (x, y) = L_x (u_0 - lyu_0) \]
\[ L_x \int_{n=0}^{\infty} dx = x \]
\[ u_2(x, y) = L_x (u_1 - lyu_1) \]
\[ L_x^{-1} = \int^x \ dx = \frac{1}{2} \ x \ 2 \]

(2) \[ L_y u(x, y) = u - L_x u \]

\[ L^{-1} y (lyu(x, y)) = L^{-1} y (u - lxu) \]

\[ u(x, y) = 1 + e^y = L^{-1} y (u - lxu) \]

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \]

\[ = \sum_{n=0}^{\infty} u_n(x, y) = 1 + e^y + L^{-1} y (u - lxu) \]

\[ u_0 + y_1 + \ldots + L^{-1} y (u_0 + y_1 + \ldots) \]

\[ u_0(x, y) = 1 + e^y \]

\[ u_{k+1}(x, y) = L^{-1} y (u_k - Lxu_k), \quad k \geq 10 \]

\[ u_1(x, y) = L^{-1} y (u_0 - lxu_0) \]

\[ L^{-1} y = \sum_{n=0}^{\infty} dy = y \]

\[ u_2(x, y) = L^{-1} y (u_1 - lxu_1) \]

\[ L^{-1} y = \sum_{n=0}^{\infty} y dy = \frac{1}{2} y^2 \]

\[ u(x, y) = u_0(x, u) + u_1(x, u) + u_2(x, y) + \ldots \]

\[ u(x, y) = e^x + e^y \]
**Example (8):**

Solve the equation:

\[ u_x + u_y = 2u \]

\[ u(x, 0) = e^x \]

\[ u(0, y) = e^y \]

**solution:**

\[ Lx = \frac{\partial}{\partial x}, \quad ly = \frac{\partial}{\partial y} \]

\[ Lxu(x, y) = 2u - lyu(x, y) \]

\[ L^{-1}x (lxu(x, y) = L^{-1}x (2u - lyu(x, y)) \]

\[ u(x, y) = e^x + L^{-1}x(2u - lyu) \]

\[ = \sum_{n=0}^{\infty} \ u_n(x, y) = e^x + L^{-1}x(2) \sum_{n=0}^{\infty} u_n - (Ly) \sum_{n=0}^{\infty} u_n \]

\[ u_0(x, y) = e^x \]

\[ u_{k+1}(x, y) = L^{-1}x(2u_k - Lyu_k), \ k > 0 \]

\[ u_1(x, y) = L^{-1}x(2u_0 - Lyu_0) \]

\[ L^{-1}x = 0^{\int x} dx = x \]

\[ u_2(x, y) = L^{-1}x(2y - lyu_1) \]

\[ L^{-1}x = \int_0^x xdx = \frac{1}{2} x^2 \]

\[ (2) lyu(x, y) = (2u - lxu) \]

\[ Ly = \frac{\partial}{\partial y} \]

\[ L^{-1}y(lxu(x, y)) = L^{-1}y(2u - lxu) \]
\[ u(x, u) = u(0, y) + L^{-1}y (2u - Lu) \]

\[ u(x, y) = e^y + L^{-1}y(2u - Lu) \]

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = e^y + L^{-1}y \left( 2 \sum_{n=0}^{\infty} u_n - Lx \sum_{n=0}^{\infty} u_n \right) \]

\[ u_0(x, y) = e^y \]

\[ u_{k+1}(x, y) = L^{-1}y(2u_2 - Lxu_k), \quad k \geq 0 \]

\[ u_1(x, y) = L^{-1}y(2u_0 - Lxu_0) \]

\[ L^{-1}y = \int_{0}^{y} dy = y \]

\[ u_2(x, y) = L^{-1}(2u_1 - lxy) \]

\[ L^{-1} = \int_{0}^{y} ydy = \frac{1}{2} + y^2 \]

\[ u(x, y) = u_0 + y + u_2 + \ldots \]

\[ u(x, y) = 2e^x + 2e^y \]

**Example (9):**

Solve the following partial differential equation:

\[ u_x + u_y + u_z = u \]

\[ u(0, y, z) = 1 + e^y + e^z \]

\[ u(x, 0, z) = 1 + e^x + e^z \]

\[ u(x, y, 0) = 1 + e^x + e^y \]

where:

\[ u = (x, y, z) = u_n(x, y, z) \]
solution:

\[ L_x u(x, y, z) = u - lyu - l_z u \]

\[ L_x = , L_y = , L_z = \]

Operator

\[ L^{-1} \] is an indefinite integrate to x using condition

\[ u(0, y, z) = 1 + e^y + e^z \]

Where

\[ u(x, y, z) = \sum_{n=0}^{\infty} u_n(x, y, z) \]

\[ = \sum_{n=0}^{\infty} u_n(x, y, z) = 1 + e^y + e^z + L^{-1}x \left( \sum_{n=0}^{\infty} u_n - Ly(\sum_{n=0}^{\infty} u_n) \right) \]

\[ u_0 + y + y_2 + ... = 1 + e^y + e^z + L^{-1}(u_0 + u_1 + u_2 + ...) \]

- \[ L^{-1}x(ly(u_0 + y + y_2 + ...)) \]

- \[ L^{-1}x(Lz(u_0 + u_1 + u_2 + ...)) \]

\[ u_0(x, y, z) = 1 + e^y + e^z \]

\[ u_1(x, y, z) = L^{-1}x(u_0 - Lyu_0 - Lzu_0) = x \]

\[ u_2(x, y, z) = L^{-1}x(u_1 - Lyu_1 - ly) = \frac{1}{2}x^2 \]

\[ u_3(x, y, z) = L^{-1}x(u_2 - Lyu_2 - Lzy) = \frac{1}{3}x^3 \]

\[ u_4(x, y, z) = L^{-1}x(u_3 - Lyu_3 - Lzu_3) \]

\[ = L^{-1}x = \int_{0}^{x} \frac{1}{3} x^3 = \frac{1}{4} x^4 \]

\[ u(x_1 y, z) = u_0 + u_1 + u_2 + ... \]

\[ u(x_1 y, z) = (1 + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + ... ) \]

\[ u(x_1 y 17) = e^x + e^y + e^z \]
Example (10):

Solve the equation:

\[ u_x - yu = 0 \]

\[ u_{(0,y)} = 1 \]

\[ L x u(x,y) = yu(x,y) \]

\[ L^{-1} L x u(x,y) = L^{-1} yu(x,y) \]

\[ \int_0^x \frac{du(s,y)}{ds} ds = [u(s,y)]_0^x = \]

\[ u(x,y) - u(0,y) = 1 \]

\[ u(x,y) = 1 + L^{-1} yu(x,y) \]

\[ u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \]

\[ \sum_{n=0}^{\infty} u_n(x,y) = 1 + L^{-1} \left( y \sum_{n=0}^{\infty} u_n(x,y) \right) \]

\[ u_0(x,y) = 1 \quad u_{k+1}(x,y) = \sum_{n=0}^{\infty} u_n(x,y) = 1 + L^{-1} \left( y \sum_{n=0}^{\infty} u_n(x,y) \right) \]
Chapter Two

The noise terms phenomenon:

2.1 Introduction:

In this chapter we will present a useful tool that will accelerate the convergence of the Adomian decomposition method.

The noise term phenomenon provides major advantages in that it demonstrates a fast convergence of the solution. It is important to note here that the noise terms phenomenon, that will be introduced in this section, may appear only or in homogeneous PDEs. In addition, this phenomenon is applicable to all in homogeneous PDEs of any order and will be used where appropriate in the coming.

In view of these remarks, the phenomenon can be drawn as follows:

1) The noise terms are defined as the identical terms with opposite signs that may appear in the components and $u_1$.

2) The noise terms appear only for specific types in homogeneous equations where as noise terms don’t appear for homogeneous equations.

3) Noise terms may appear if the exact solution is part of the zeroth components $u_0$.

4) Verification that the remaining non canceled terms satisfy the equation is necessary and essential.

2.2 Some examples:

Example (1):

Use the decomposition method and the noise terms phenomenon to solve the following in homogeneous PDE:

$$u_x + u_y = \sinh x + \sinh y$$

$$u(x, 0) = 1 + \cosh x$$

$$u(0, y) = 1 + \cosh y$$

solution:

$$Lyu = (\sinh x + \sinh y) - Lxu$$

$$L^{-1}y(lyu) = L^{-1}y(\sinh x + \sinh y) - L^{-1}(lxu)$$

$$u(x, y) = 1 + \sinh x + L^{-1}yu(\sinh x + \sinh y)$$

$$- L^{-1}yu(lxu)$$
\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \]

\[ = \sum_{n=0}^{\infty} u_n(x, y) = 1 + \sinh x + L^{-1}yu(\sinh x + \sinh y) - L^{-1}yu(lx( \sum_{n=0}^{\infty} u_n(x, y) ) \]

\[ u_0(x, y) = 1 + \sinh x + (\sinh x + \cosh y) \]

\[ u_1(x, y) = -L^{-1}uy(lxu_0) \]

\[ = -\cosh x - (y \coch x - \sinh y) \]

\[ u_2(x, y) = -L^{-1}yu(lxu) \]

\[ \sinh x + y \sinh x + \cosh y \]

\[ u(x, y) = u_0 + u_1 + u_2 + \ldots \]

\[ u(x, y) = 1 + \sinh x + (\sinh x + \cosh y) \]

\[ -\cosh x - y \cosh x - \sinh y \]

\[ + \sinh x + y \sinh x + \cosh y \]

\[ u(x, y) = 1 + \sinh x + \cosh y \]

(2) \[ Lxu = (\sinh x + \sinh y) - Lyu \]

\[ L^{-1}x(lxu) = L^{-1}x(\sinh x + \sinh y) - L^{-1}xu(lyu) \]

\[ u(x, y) = 1 + \sinh y + (\cosh x + \sinh y) - L^{-1}xu(lyu) \]

\[ u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \]

\[ = \sum_{n=0}^{\infty} u_n(x, y) = 1 + \sinh y + (\cosh x + \sinh y) - L^{-1}x(ly( \sum_{n=0}^{\infty} u_2 ) \]

\[ u_0(x, y) = 1 + \sinh y + (\cosh x + \sinh y) \]
\[ u_1 (x, y) = - L^{-1}x (L_y u_0) \]
\[ = - \cosh y - (\sinh x - x \cosh y) \]
\[ u_2(x, y) = - L^{-1}x (L_y u_1) \]
\[ = \sinh y + (\cosh x + x \sinh y) \]
\[ u(x, y) = u_0 + y + u_2 + ... \]
\[ u(x, y) = 1 + \sinh y + \cosh x \]

**Example (2):**

Solve the following PDE:

\[ u_x - u_y = \cos x + \sin y \]
\[ u(x,0) = 1 + \sin x \]
\[ u(0,y) = \cos y \]

**solution:**

\[ Lyu = (\cos x + \sin y) + Lxu \]
\[ L^{-1}(lyu) = L^{-1}(\cos x + \sin y) + L^{-1}(L_xu) \]
\[ u(x,y) = 1 + \cos x + L^{-1}y(\cos x + \sin y) + L^{-1}y(L_x \sum_{n=0}^{\infty} u_n(x, y)) \]
\[ = \sum_{n=0}^{\infty} u_n = 1 + \cos x + L^{-1}y(\cos x + \sin y) + L^{-1}y(L_x \sum_{n=0}^{\infty} u_n(x, y)) \]
\[ u_0(x, y) = 1 + \cos x - \sin x - \cos y \]
\[ u_1(x, y) = L^{-1}y(L_x u_0) \]
\[ = - \sin x + \cos x + \sin y \]
\[ u_2(x, y) = L^{-1}(L_xu_1) \]
\[ = - \cos x - \sin x + \cos y \]
\[ u(x, y) = u_0 + u_1 + u_2 + \ldots \]

\[ u(x, y) = 1 + \cos x - (\sin x - \cos y) - (\sin x + \cos y + \sin y) \]

\[ - (\cos x - \sin x + \cos y) \]

\[ u(x, y) = 1 + \sin x + \cos y \]

**Example (3):**

Use the noise terms phenomenon to solve the following PDE:

\[ u_x + u_y = x^2 + 4xy + y^2 \]

\[ u(0, y) = 0 \]

\[ u(x, 0) = 0 \]

**Solution:**

We first rewrite in homogeneous in an operator.

\[ L_x u = x^2 + 4xy - Lyu \]

\[ L^{-1}x(l_x u) = L^{-1}x(x^2 + 4x y) - L^{-1}x (L_y u) \]

\[ u(x, y) = 1/3 x^3 + 2x^2 y + y^2 - L^{-1}x(L_y u(x, y)) \]

\[ u_0(x, y) = 1/3 x^3 + 2x^2 y + xy^2 \]

\[ u_1(x, y) = - L^{-1}x(yu_0) \]

\[ = - x^2 y - 2/3 x^3 \]

\[ u_2(x, y) = - L^{-1}x(-x^2) = 1/3 x^3 \]

\[ u_k(x, y) = 0, k \geq 3 \]

\[ u(x, y) = u_0 + u_1 + u_2 + \ldots \]

\[ u(x, y) = 1/3 x^3 + 2x^2 + xy^2 - 2/3 x^3 + 1/3 x^3 \]

\[ u(x, y) = xy^2 + x^2 y = \]
chapter Three
Adomian decomposition method
For Solving PDES with Variable Coefficient

3.1 Lap Lace Equations:
Analysis of the Adomian decomposition method to illustrate the basic concepts of the Adomian decomposition method.

For solving the lap lace equation, first we rewrite it in the following form:
\[ L_{xx}u(x, y) + L_{yy}u(x, y) = 0 \] (3.1)

Where the notions
\[
L_{xx} = \frac{\partial^2}{\partial x^2} \text{ and } L_{yy} = \frac{\partial^2}{\partial y^2}
\]

Operating with inverse operator both sides E2(a)
We obtain
\[
L^{-1}_{xx} = \int_0^x \int_0^y dt dy
\]

Consider equation (3.1) with the bounder y condition.

Example (1):

\[
L^{-1}_{xx}(x, y) = -L^{-1}_{yy}(x, y)
\]
solution:

where \( u_0(x, y) = x \cos y \)

\[
u_1(x, y) = - L^{-1}xx(lyy u_0(x, y))
\]

\[
L^{-1}xx = \int_0^x \int_0^x dt \, dt
\]

\[= \frac{1}{3} x^3 \cos y \]

\[
u_2(x, t) = - L^{-1}xx(lyy u_1(x, y))
\]

\[= \frac{1}{5} x^5 \cos y \]

\[
u_3(x, t) = - L^{-1}xx(lyy u_2(x, y))
\]

\[= \frac{1}{7} x^7 \cos y \]

\[
u(x, y) = u_0 + u_1 + u_2 + \ldots
\]

\[
u(x, y) = \cos y (x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \frac{1}{7} x^7)
\]

so the exact solution:

\[
u(x, y) = \sinh x \cos y
\]

Example (2):

Consider equation (3.1) with the boundary condition:

\[
u_2(x, y) = \sin y
\]

solution:

where

\[
u_0(x, y) = \sin y
\]

\[
u_1(x, y) = - L^{-1}xx(lyy u_0(x, y))
\]

\[
L^{-1}xx = \int_0^x \int_0^x dt \, dt
\]

\[= \frac{1}{2} x^2 \sin y \]

\[
u_2(x, t) = L^{-1}xx(lyy u_1(x, y))
\]
\[ u_3(x, t) = -L^{-1}xx(lyyu_2(x, y)) = \frac{1}{4} x^4 \sin y \]

\[ u(x, y) = u_0 + u_1 + u_2 + u_3 + \ldots \]

\[ u(x, y) = \sin y(1 + \frac{1}{2} x^2 + \frac{1}{4} x^4 + \frac{1}{6} x^6) \]

so the exact solution is:

\[ u(x, y) = \cos x \sin y \]

**Example (3):**

Consider equation (3.1) with the boundary condition:

\[ u_2(x, y) = \cos 2x \]

**solution:**

where

\[ u_0(x, y) = \cos 2x \]

\[ u_{n+1}(x, y) = -L^{-1}yy(lxxu_0(x, y)) \]

\[ u_1(x, y) = -L^{-1}yy(lxy(x, y)) \]

\[ L^{-1}yy = \int \int dt \]

\[ = 2y^2 \cos 2x \]

\[ u_2(x, t) = -L^{-1}yy(Lxxy(x, y)) \]

\[ 2/3 \ y^4 \cos 2x \]

\[ u_3(x, t) = -L^{-1}yy(lxxu_2(x, y)) \]

\[ = 4/54 \ y^6 \cos 2x \]

\[ u(x, y) = u_0 + u_1 + u_2 + u_3 + \ldots \]

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\[ u(x, y) = \cos 2x (1 + 2y^2 + \frac{2}{3}y^4 + \frac{4}{54}y^6) \]
	herefore the exact solution in closed form will be:

\[ u(x, y) = c + \cos 2x \cos h 2y \]

**Example (4):**

Consider equation (3.1) with the boundary condition:
\[ u_0(x, y) = y \cos x \]

**solution:**

\[ u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) + u_3 + \ldots \]

\[ u_0(x, y) = y \cos x \]

\[ u_1(x, y) = -L^{-1}yy(Lxxu_0(x, y)) \]
\[ = \frac{1}{3}y^3 \cos x \]

\[ u_2(x, y) = -L^{-1}yy(Lxxu_1(x, y)) \]
\[ = \frac{1}{5}y^5 \cos x \]

\[ u_3(x, y) = -L^{-1}yy(Lxxu_2(x, y)) \]
\[ = \frac{1}{7}y^7 \cos x \]

\[ u(x, y) = u_0 + u_1 + u_2 + u_3 + \ldots \]

\[ u(x, y) = \cos x (1 + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7) \]

therefore the solution in closed form
\[ u(x, y) = c + \cos x \sinh y \]
3.2 Laplace transformation

To illustrate the basic idea of this method.

We considered the general form of one dimensional non-homogeneous partial differential equations with variable coefficient of the form.

\[
\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + \psi (x,t)
\]

and

\[
\frac{\partial^2 u}{\partial t^2} = \mu \frac{\partial^2 u}{\partial x^2} + \psi (x,t)
\]

Example (1):

let us Consider the one dimensional non-homogeneous problem:

\[
\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + u - e^{-x} (1 + 2t) = 0
\]

Boundary condition

\[
u(0,t) = t
\]

\[
u_x (0, t) = e^{-2} - t \quad \text{initial condition}
\]

solution:

\[
u_{xx} = u - e^{-x}(1 + 2t) - ut
\]

\[
Lu_{xx} = u - e^{-x}(1 + 2t) - ut
\]

\[
L^{-1}_{xx}(lu) = L^{-1}_{xx}(u - e^{-x}(1 + 2t) - L^{-1}_{xx}(ut))
\]

\[
u(x,t) = x - e^{-x}(1 + 2t) - L^{-1}_{xx}(u_2)
\]

\[
\nu(x,t) = \sum_{n=0}^{\infty} u_n(x,t)
\]

\[
= \sum_{n=0}^{\infty} u_n(x,t) = x - e^{-x}(1 + 2t) - L^{-1}_{xx}(\sum_{n=0}^{\infty} u_n(x,t))
\]
\[ u_0(x,t) = t \, e^{-x} + x - e^{-x} (1 + 2t) \]
\[ u_1(x,t) = - L^{-l}_{xx}(u_0t) \]
\[ = \frac{1}{6} x^2 + e^{-x} (1 + 2t) - x (1 + 2t) - (1 + 2t) \]
\[ u_2(x,t) = - L^{-l}_{xx}(u, t) \]
\[ = \frac{1}{120} x^5 + e^{-x} (1 + 2t) + \frac{1}{3} x^{-3} (1 + 2t) + \frac{1}{2} x^2 (1 + 2t) \]
\[ u(x,t) = u_0 + u_1 + u_2 + \ldots \]
\[ u(x,t) = t \, e^{-x} \]
\[ u_{xx} = u - e^{-x} (1 + 2t) - ut \]
\[ Lu = u - e^{-x} (1 + 2t) - ut \]
\[ L^{-l}_{xx}(u) = L^{-l}_{xx}(u - e^{-x} (1 + 2t)) - L^{-l}_{xx}(ut) \]
\[ u(x,t) = y \, e^{-t} - t + x - e^{-x} (1 + 2t) - L^{-l}_{xx}(ut) \]
\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \]
\[ = \sum_{n=0}^{\infty} u_n(x,t) = xe^{-t} - t + x - e^{-x} (1 + 2t) - L^{-l}_{xx}(\sum_{n=0}^{\infty} u_n(x,t)) \]
\[ u_0(x,t) = y \, e^{-t} - t + y - e^{-x} (1 + 2t) \]
\[ u_1(x,t) = - L^{-l}(u_0) t \]
\[ = - \frac{1}{6} x^3 - e^{-x} (1 + 2t) - x (1 + 2t) - (1 + 2t) \]
\[ u_2(x,t) = - L^{-l}_{xx}(u, t) \]
\[ = \frac{1}{120} x^5 - e^{-x} (1 + 2t) + \frac{1}{3} x^{-3} (1 + 2t) + \frac{1}{2} x^2 (1 + 2t) \]
\[ + \frac{1}{2} x^2 (1 + 2t) + x (1 + 2t) - (1 + 2t) \]
\[ 38 \]
\[ u(x,t) = u_0 + u_1 + u_2 + \ldots \]
\[ u(x,t) = x e^t \]
\[ u(x,t) = t e^{-x} + x e^t \]

**Example (2):**

Let us consider the problem:

\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u = 0 \]

\[ u(0,2) = \cos(t) \]
\[ u_x(0,t) = 1 \]

Initial condition
\[ u_t(x,0) = 0 \]

**Solution:**

\[ u_{xx} = u - u_t \]
\[ L_{xx} = \frac{d}{dx^2} \]
\[ L^{-1}_{xx} = \int_0^x \int_0^x dx dx \]

\[ Lu = (u - u_t) \]
\[ L^{-1}_{xx} l(u) = L^{-1}_{xx} (u - u_t) \]
\[ U(x,t) = x + \cosh(t) - L^{-1}_{xx} (u - u_t) \]

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \]

\[ \sum_{n=0}^{\infty} u_n(x,t) = y + \cosh(t) - L^{-1}_{xx} \left( \sum_{n=0}^{\infty} u_n(x,t) \right) \]
\[ u_0(x,t) = x + \cosh(t) \]
\[ u_1(x,t) - L^{-1}_{xx} (u_0 - u_0 t) \]
\[ = - \frac{1}{6} x^3 \]
\[ u_2(x, t) = \cdot L^{-1}xx(u_1 - u, tt) \]
\[ = \frac{1}{120} x^5 \]
\[ u(x, t) = u_0 + u_1 + u_2 + \ldots \]
\[ u_1(x, t) = \cosh(t) + x - \frac{1}{6} x^3 + \frac{1}{120} x^5 + \ldots \]
\[ u(x, t) = \cosh(t) + \sin(t) \]

**Example (3):**
Solve the equation:
\[ \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + e^{-x} (\cos(t) - \sin(t)) \]

*Boundary condition*
\[ u(0, t) = \sin(t) \]
\[ u(0, z) = \sin(t) \]
\[ u(0, t) = \frac{1 + \sin(t)}{e} \]

*Initial condition*
\[ u(x, 0) = x \]

*solution:*
\[ u_{xx} = e^{-x}(\cos(t) - \sin(t)) - ut \]
\[ lu = e^{-x}(\cos(t) - \sin(t)) - ut \]
\[ L^{-1}_{xx}l(u) = L^{-1}_{xx}(e^{-x}(\cos(t) - \sin(t)) - L^{-1}_{xx}(ut)) \]
\[ u(x, t) = x + e^{-x}(\sin(t) + L^{-1}_{xx}(e^{-x}(\cos(t) - \sin(t)) - \sin(t) - L^{-1}_{xx}(ut)) \]
\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \]
\[
\sum_{n=0}^{\infty} u_n(x, t) = x + e^{-x} (\sin(t) + L^{-1}_{xx}(e^{-x}(\cos(t) - \sin(t)) -
\sum_{n=0}^{\infty} u_n(x, t)
\]

\[u_0(x, t) = x + e^{-x}(\sin(t))\]

\[u_1(x, t) = -L^{-1}_{xx}(u_0 t)\]

\[= - e^{-x}(\cos(t) - \sin(t) - x(\cos(t) - \sin(t))\]

\[u_2(x, t) = -L^{-1}_{xx}(ut)\]

\[= \frac{1}{6} x^3 (\cos(t) - \sin(t)) + \frac{1}{2} x^2 (\cos(t) - \sin(t)\]

\[u(x, t) = u_0 + u_1 + u_2 + ...\]

\[u(x, t) = x + e^{-x} \sin(t)\]

(2)

\[u(1, t) = \frac{1 + \sin(t)}{e}\]

\[lu = e^{-x}(\cos(t) - \sin(t) - ut\]

\[L^{-1}_{xx} l(u) = L^{-1}_{xx}(e^{-x}\cos(t) - \sin(t) - L^{-1}_{xx}(ut)\]

\[u(x, t) = x + e^{-x} \sin(t) + L^{-1}_{xx}(e^{-x}(\cos(t) - \sin(t) - L^{-1}_{xx}(ut)\]

\[= (x, t) = \sum_{n=0}^{\infty} u_n(x, t)\]

\[= \sum_{n=0}^{\infty} u_n(x, t) = x + e^{-x} \sin(t) + L^{-1}_{xx}(e^{-x}(\cos(t) - \sin(t)) - L^{-1}_{xx} \sum_{n=0}^{\infty} u_n(x, t)\]

\[u_0(x, t) = x + e^{-x} \sin(t)\]

\[u_1(x, t) = -L^{-1}_{xx}(u_0 t)\]

\[= - e^{-x}(\cos(t) - \sin(t) - (\cos(t) - \sin(t))\]
\[ u_2(x,t) = - L^t x x (u_t) \]

\[ = e^x (\cos(t) - \sin(t) - x (\cos(t) - \sin(t)) - (\cos(t) - \sin(t)) \]

\[ u(x,t) = u_0 + u_1 + u_2 + \ldots \]

\[ u(x,t) x + e^{-x} \sin(t) \]

\[ \therefore u(x,t) = y + e^{-x} \sin(t) \]
Conclusion and Recommends

We will present useful tool that will accelerate the convergence of the Adomian decomposition method the noise term phenomenon provides a major advantages in that it demonstrates a fast convergence of the solution it is important to note there that noise terms phenomenon that will be introduced for inhomogeneous PDES in addition, this phenomenon is applicable to all inhomogeneous PDES of isst first order and will be used where appropriate in the coming. We considered the general form of one dimensional non-homogenous partial differential; equation with variable coefficient. The A domino decomposition method is implemented to solve non-homogenous partial differential equations with a variable coefficient and decomposition partial with the exact solution.

This way is modern and has a very high ability to solve the partial differential equations. In the first order, it will have a promising results in the future. We recommend to use this way in solving the partial differential equations in a fast, direct, easy and effective method. Also it has a high technical ability to solve the partial differential equations in the second order and in the high order also. This way leads to a solution in fast, direct, easy and effective.
References

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