Differential Transformation Method for Solving Linear and Non Linear Differential Equations

Maysaa Mohammed Elamin Eltyeb Tyfoor

B.E.d in Mathematic, University of Omdurman Islamic (2005)
Postgraduate Diploma in Mathematics, University of Gezira(2014)

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Department of Mathematics
Faculty of Mathematical and Computer Sciences

May, 2016
Differential Transformation Method for Solving Linear and Non Linear Differential Equations

Maysaa Mohammed Elamin Eltyeb Tyfoor

Supervisor Committee:

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<tr>
<td>Dr. Abdulla habila Ali</td>
<td>Main Supervisor</td>
<td></td>
</tr>
<tr>
<td>Dr. Mohammed Elnaiem Ahmed Alnayer</td>
<td>Co-supervisor</td>
<td></td>
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Date : May,2016
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Examination Committee:

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<tr>
<td>Dr. Abdulla habila Ali</td>
<td>Chair Person</td>
<td></td>
</tr>
<tr>
<td>Dr. Eltiyeb Mohammed Khair Edrees</td>
<td>External Examiner</td>
<td></td>
</tr>
<tr>
<td>Dr. Saad Eldin Mohamed Saad Eldin</td>
<td>Internal Examiner</td>
<td></td>
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Date of Examination: 30/ May/ 2016
Declaration

This to certify that I had completed my thesis by myself and using references without taking and information from others. This these was taken under supervision of advisors.

Name: Maysaa Mohammed Elamin EltyebTyfoor
Dedication

I dedicate this work for my lovely parents and my brothers and my sisters and husband.
Acknowledgements

I would like to thank my supervisor Dr. Abdulla habila. For his supervision, advice and encouragement during this research. My gratitude is also for my co-supervisor Dr. Mohammed Elnaiem al-nayer mentored me through this research, and for her kind.

Thanks to everyone who helped I but I forgot to mention in these short lines.
Differential Transformation Method for Solving Linear and Non Linear Differential Equations

Maysaa Mohammed ElaminEltyebTyfoor

Abstract

Differential Transformation Method (DTM) is a method in which we use directly affects the differential law are a function of differentiation know Taylor series at a certain point. We used ito find the numerical solution of the linear ordinary differential equations and nonlinear differential equations, (homogeneous or inhomogeneous).

Using DTM to solve the Lane-Emden equations as singular initial value problems is introduced in this study. We propose the generalization of DTM to solve higher order of linear boundary value problem. In our study, we generalized the method so that one can solve n-th order boundary value problems with m-th order linear differential equation for m>n, <n or m=n. To illustrate the accuracy of the proposed method, we provide several numerical examples and we compare the results with the exact solutions. The comparisons demonstrate the proposed method has high accuracy.

We study DTM is applied for solving system of Linear Differential Equations and non-linear differential equations. The approximate solution of the equation is calculated in the form of series with easily computable component. Differential transform method is proposed for the closed form solution linear of and nonlinear stiff system. first we apply DTM to find the series solution which can be easily converted into exact solution. the obtained result confirm that DTM is very easy, effective and convenient. Using DTM second order complex differential equation system was solved, firstly we separated real and imagined parts. After that, by using two dimensional differential transform we obtained real and imagined parts of solutions. We present a numeric-analytic solution of the well known Michaelis–Menten nonlinear biochemical reaction system based on differential transformation method (DTM). If we have two- Dimensional Differential Transform Method (2D DTM) is applied to obtaining the analytic solution of the two-dimensional non-linear wave equation. The results demonstrate reliability and efficiency of this method for such problems. From previous studies show that the differential transformation method is a powerful method to solve many of problems in ordinary differential equations and partial differential equations.

In this study, we obtained the results of this method are in clearly series.
طريقة التحويل التفاضلي لحل المعادلات التفاضلية الخطية والغير خطية

ميساء محمد الأمين الطيب طيفور

ملخص الدراسة

طريقة التحويل التفاضلي هي طريقة مباشرة نستخدم فيها قانون تفاضلي يتأثر على دالة قابلة للتفاضل معروفة بيفكك تأثير عند نقطة معينة. نستخدمها لإيجاد الحل العادي للمعادلات التفاضلية الخطية والمعادلات التفاضلية غير الخطية، (متجانسة أو غير متجانسة). كما نستخدم DTM لحل معادلات لين. Meanwhile كما هو معروض مشكلة القيمة الأولية فريدة في هذه الدراسة. نستخدم DTM حال مسألة الرتب العليا القيمة الحدية الخطية، ونعمل الطرقية بحيث يمكن للمرء أن يحل مسألة قيمة حدية ألبية. يتم تطبيق DTM من أجل حل نظام من المعادلات التفاضلية الخطية والمعادلات التفاضلية غير الخطية. وحساب حل تقريبي من المعادلة. يتم حل بعض الأنظمة الخطية (أو غير الخطية) للمعادلات التفاضلية كشيئًا عديدًا، نطبق DTM للعثور على الحل في صورة متسلسلة والتي يمكن تحويلها بسهولة إلى الحل الصحيح، وهي سهلة جدا وفعالة 

تم حل نظام المعادلات التفاضلية المركزية من الدرجة الثانية، أولا تم حل الأجزاء الحقيقية والتخيلية ثم باستخدام DTM حصلنا على أجزاء الحقيقية والتخيلية من الحلول. نقدم الحلول الرقمية التحليلية للميكانيك. ميتينت غير الخطية نظام رد فعل كيميائي حيوي مشهور تستخدم على طريقة التحول التفاضلي (DTM) النتائج الرقمية التي تم الحصول عليها من DTM هي في وضع تام في صورة متسلسلة، وتشير دراسات سابقة إلى أن طريقة التحويل التفاضلي هو سهلة قوية لحل كثير من المسائل في المعادلات التفاضلية العادية والمعادلات التفاضلية الجزئية وغيرها بصورة سلسة.
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Chapter One

Differential transformation for solving of linear and non-linear ODEF

Introduction:

A variety of methods, exact, approximate and purely numerical are available for the solution of differential equations. Most of these methods are computationally intensive because they are trial-and-error in nature, or need complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations. The concept of differential transformation was first proposed by Zhou [11] in 1986 [12-15] and it was applied to solve linear and non-linear initial value problems in electric circuit analysis. This method constructs a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The Differential transformation method is very effective and powerful for solving various kinds of differential boundary value problems [16], to differential-algebraic equations [17], to the KdV and MKdV equations [18], to the Schrödinger equations [19] to fractional differential equations [20] and to the Riccati differential equation [21]. Jang et al. [22] introduced the application of the concept of the differential transformation of fixed grid size to approximate solutions of linear and non-linear initial value problems. Hassan [23] applied the differential transformation technique of fixed grid size to solve the higher-order initial value problems. The transformation method can be used to evaluate the approximating solution by the finite Taylor series and by an iteration procedure described by the transformed equations obtained from the original equation using the operations of differential transformation. The main advantage of this method is that it can be applied directly to nonlinear ODEs without requiring linearization, discretization or perturbation. Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate [24].
In this paper, we shall apply DTM to find the approximate analytical solution of the first, second and third order linear ordinary differential equation. Comparisons with the exact solution will be performed.

**(1.1) A New Algorithm For Solving Linear Ordinary Differential Equation**

To find the approximate analytical solution of the first, second and third order linear ordinary differential equation. Comparisons with the exact solution with be performed. We shall apply differential transformation method to find the solution

An arbitrary function f(x) can be expanded in Taylor series about a point X=0 as:

\[
 f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left( \frac{d^k f}{dx^k} \right)_{x=0}
\]

(1.1)

Taylor series \( f(x) = f(a) + (x-a) \frac{d f}{dx}(a) + \frac{(x-a)^2}{2!} \frac{d^2 f}{dx^2}(a) + \ldots \) at \( x=a \)

If \( x = 0 \) it become maclaurin series

\[
 f(x) = f(0) + x \frac{d f}{dx}(0) + \frac{x^2}{2!} \frac{d^2 f}{dx^2}(0) + \ldots
\]

The differential transformation of \( f(x) \) is defined as:

\[
 F(x) = \frac{1}{k!} \left( \frac{d^k f}{dx^k} \right)_{x=0}
\]

(1.2)

Then the inverse differential transform is

\[
 f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left( \frac{d^k f}{dx^k} \right)_{x=0}
\]
Table (1.1.1): The fundamental operation of Differential Transformation method (DTM):

<table>
<thead>
<tr>
<th>Original function</th>
<th>Transformed function</th>
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<tbody>
<tr>
<td>$y(x)=h(x)\pm g(x)$</td>
<td>$Y(k)=H(k)\pm G(k)$</td>
</tr>
<tr>
<td>$y(x)=\alpha g(x)$</td>
<td>$Y(k)=\alpha G(k)$</td>
</tr>
<tr>
<td>$y(x)=\frac{dg(x)}{dx}$</td>
<td>$Y(k)=(k+1)G(k+1)$</td>
</tr>
<tr>
<td>$y(x)=\frac{d^2g(x)}{dx^2}$</td>
<td>$Y(k)=(k+1)(k+2)G(k+2)$</td>
</tr>
<tr>
<td>$y(x)=\frac{d^m g(x)}{dx^m}$</td>
<td>$Y(k)=(k+1)(k+2)\ldots(k+m)G(k+m)$</td>
</tr>
<tr>
<td>$y(x)=1$</td>
<td>$Y(k)=\delta(k)$</td>
</tr>
<tr>
<td>$y(x)=x$</td>
<td>$Y(k)=\delta(k-1)$</td>
</tr>
<tr>
<td>$y(x)=x^m$</td>
<td>$Y(k)=\delta(k-m)={\begin{cases} 1 &amp; k = m \ 0 &amp; k \neq m \end{cases}}$</td>
</tr>
<tr>
<td>$y(x)=g(x)h(x)$</td>
<td>$Y(k)=\sum_{m=0}^{k} H(m)G(k-m)$</td>
</tr>
<tr>
<td>$y(x)=e^{\omega x}$</td>
<td>$Y(k)=\frac{\omega^k}{k!}$</td>
</tr>
<tr>
<td>$y(x)=(1+x)^m$</td>
<td>$Y(k)=\frac{m(m-1)\ldots(m-k+1)}{k!}$</td>
</tr>
<tr>
<td>$y(x)=\sin(wx+\alpha)$</td>
<td>$Y(k)=\frac{w^k}{k!} \sin\left(\frac{k\pi}{2} + \alpha\right)$</td>
</tr>
<tr>
<td>$y(x)=\cos(wx+\alpha)$</td>
<td>$Y(k)=\frac{w^k}{k!} \cos\left(\frac{k\pi}{2} + \alpha\right)$</td>
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First order ODES:

**Example (1.1):** To solve the following first order homogenous ODE

$$\dot{u} - 2xu = 0\ ,\ u(0) = 1 \quad (1.3)$$

By using the operation in table (1.1.1) above we obtained following repeating relation:

$$\frac{(k+1)!}{(k+1)!} \frac{d^{k+1} u}{dx^{k+1}} = 2 \sum_{m=0}^{k} \delta(m-1)U(k-m)$$

$$U(k+1) = \frac{1}{k+1} \left[ 2 \sum_{m=0}^{k} \delta(m-1)U(k-m) \right] \quad (1.4)$$

By taking DTM for The initial condition $u(0) = 1$ become $U(0) = 1$ from equation (1.4).
Therefore, the closed form solution can be easily written as

\[ u(x) = \sum_{m=0}^{k} U(k)x^k = x + x^2 + \frac{1}{2!}x^3 + \cdots + \frac{1}{n!}x^{n+1} = xe^x \]

for checking the salutation in equation (1.3)

\[ 2xe^{x^2} - 2xe^{x^2} = 0 \]

**Example (1.2):** Deem the first order non-homogeneous ODE

\[ \dot{u} - u = e^x \quad u(0) = 0 \]  \hspace{1cm} (1.5)

By using the fundamental operations of differential transformation method in table 1 it's can be

\[ U(k+1) = \frac{1}{k+1}[U(k) + \frac{1}{k!}] \]  \hspace{1cm} (1.6)

From the initial condition \( U(0) = 0 \) we have \( u(0) = 0 \) and from equation (1.6) above we have:

\[ U(1) = 1 \]
\[ U(2) = 1 \]
\[ U(3) = \frac{1}{2} = \frac{1}{2!} \]
\[ U(4) = \frac{1}{6} = \frac{1}{3!} \]
\[ U(5) = \frac{1}{24} = \frac{1}{4!} \]

Therefore, the solution can be easily written as

\[ u(x) = \sum_{m=0}^{k} x^k U(k) = x + x^2 + \frac{1}{2!}x^3 + \cdots + \frac{1}{n!}x^{n+1} = x[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n] = xe^x \]  \hspace{1cm} (1.7)
for checking the salutation in equation(1.5)

\[ xe^x + e^x - xe^x = e^x \]

**Second Order ODEs:**

**Example (1.3):** consider the following second order homogenous ODE

\[ \dot{u} + u = 0 \]  (1.8)

Subject to the initial condition

\[ u(0) = 1 \quad \dot{u}(0) = 0 \]  (1.9)

Taking the differential transform of (1.8) leads to

\[ U(k + 2) = \frac{-1}{(k+1)(k+2)}[U(k)] \]  (1.10)

From the initial conditions given by equation (1.7) we have

\[ U(0) = 1 \quad U(1) = 1 \]  (1.11)

By (9) and (10) the results was listed as follows

\[ U(2) = \frac{-1}{2} = \frac{-1}{2!} \]

\[ U(3) = \frac{-1}{6} = \frac{-1}{3!} \]

\[ U(4) = \frac{1}{24} = \frac{1}{4!} \]

\[ U(5) = \frac{1}{120} = \frac{1}{5!} \]

\[ U(6) = \frac{-1}{20} = \frac{-1}{6!} \]

\[ U(7) = \frac{-1}{5040} = \frac{-1}{7!} \]  (1.12)

Therefore, the finally solution can be easily written as

\[ u(x) = \sum_{m=0}^{k} x^k U(k) \]

\[ = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots \]

\[ = (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots) \]

\[ + (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots) \]
\[ = \sin x + \cos x \quad \text{(1.13)} \]

for checking the salutation in equation (1.8)

\[-\sin x - \cos x + \sin x + \cos x + 0 \quad \text{Type equation here.}\]

**The Third Order ODEs**

**Example (1.4):** Deem the following third-order initial value problem

\[ \dot{u} + 2\ddot{u} - \dot{u} - 2u = e^x \quad \text{(1.14)} \]

With initial conditions

\[ u(0) = 1, \quad \dot{u}(0) = 2, \quad \ddot{u}(0) = 0, \quad \text{(1.15)} \]

Taking the (DTM) of (1.14), leads to

\[ U(k+3) = \frac{1}{(k+1)(k+2)(k+3)} \times \]

\[ [(k+1)U(k+1)-2(k+2)U(k+2)+2U(k)+\frac{1}{k!}] \quad \text{(1.16)} \]

Therefore, the exactly solution can be easily written as:

\[ U(0) = 1 \quad \text{(1.17)} \]

\[ U(1) = 2 \quad \text{(1.18)} \]

\[ U(2) = 0 \quad \text{(1.19)} \]

Substituting equation (1.16) into Eqs. (1.17) and (1.19) and by recursive method, the results are listed as follows

\[ U(3) = \frac{5}{6} \]

\[ U(4) = \frac{5}{24} \]

\[ U(5) = \frac{2}{15} \]

\[ U(6) = \frac{13}{360} \]

\[ U(7) = \frac{59}{5040} \quad \text{(1.20)} \]

\[ u(x) = 1 + 2x + \frac{5}{6}x^3 - \frac{5}{24}x^4 + \frac{2}{15}x^5 - \frac{13}{360}x^6 + \frac{59}{5040}x^7 - \frac{3}{13440}x^8 \quad \text{(1.21)} \]

We shall note that, in this differential equation we need the higher order of DTM to get an accurate solution.
(1.2) Solution of Non-Linear Differential Equation by Using Differential Transform Method

1- Analysis of differential transform

We will introduce the Liouville and differential transform of non-linear functions.

1- Exponential non linearity:

\[ f(x) = e^{ay} \]

From the definition of transform:

\[ F(0) = \left[ e^{ay(x)} \right]_{x=0} = e^{ay(0)} = e^{ay(0)} \]  

(1.23)

By differentiating \( f(y) = e^{ay} \) with respect to \( x \), we get:

\[ \frac{df(x)}{dx} = e^{ay} \frac{dy(x)}{dx} = a f(y) \frac{dy(x)}{dx} \]

(1.24)

Application of the differential transform to Eq. (1.23) gives:

\[ (k+1) F(k+1) = a \sum_{m=0}^{k} (m+1) Y(m+1) F(k-m) \]  

(1.25)

Replacing \( k+1 \) by \( k \) gives:

\[ F(k) = a \sum_{m=0}^{k} ((m+1)/k) Y (m+1) F(k-1-m) \]  

(1.26)

From above we obtain the recursive relation:

\[ F(k) = \begin{cases} e^{ay(0)} & k = 0 \\ a \sum_{m=0}^{k} ((m+1)/k) Y (m+1) F(k-1-m) & k \geq 0 \end{cases} \]  

(1.27)

2- Logarithmic non-linearity:

\[ F(y) = \ln (a + by), \quad a + by > 0 \]

Differentiating \( f(y) \) with respect to \( x \) we get:

\[ \frac{df(y)}{dx} = \frac{b}{a+by} \frac{dy(x)}{dx} \quad \text{or} \quad a \frac{df(y)}{dx} = b \left[ \frac{dy(x)}{dx} - y \frac{df(y)}{dx} \right] \]

(1.28)

By definition of transform:

\[ F(0) = \ln [a + by(x)]_{x=0} = \ln [a + by(0)] = \ln [a + b Y(0)] \]

(1.29)

Take the differential transform of equation (1.28) to get:
\[ a \ F(k+1) = b \left[ Y(k+1) - \sum_{m=0}^{k} (m+1)/(k+1) \ F(m+1) \ Y(k-m) \right] \]  \quad (1.30)

Replacing \( k+1 \) by \( k \) yields:

\[ a \ F(k) = b \left[ Y(k) - \sum_{m=0}^{k-1} (m+1)/(k+1) \ F(m+1) \ Y(k-1-m) \right] \quad k \geq 1 \]  \quad (1.31)

Put \( k=1 \) in Eq. (1.31) we get:

\[ F(1) = \frac{b}{a+by(0)} \ Y \]  \quad (1.32)

For \( k \geq 1 \) we can be written as:

\[ F(k) = \frac{b}{a+by(0)} \left[ Y(k) - \sum_{m=0}^{k-1} (m+1)/(k+1) \ F(m+1) \ Y(k-1-m) \right] \]  \quad (1.33)

Thus the recursive relation is:

\[
F(k) = \begin{cases} 
\ln \left[ a + b \ Y(0) \right] & K = 0 \\
\frac{b}{a+by(0)} \ Y(1) & k = 1 \\
\frac{b}{a+by(0)} \left[ Y(k) - \sum_{m=0}^{k-2} (m+1)/(k+1) \ F(m+1) \ Y(k-1-m) \right] & k \geq 2 
\end{cases}
\]

3- Application:

Example (1.5): Solve

\[ f'(v) = \left[ f(r) \right]^2 \quad f(0) = 1 \]  \quad (1.34)

Solution:

\[ f' = f^2 \]  \quad (1.35)

\[ f(0) = 1 \]  \quad (1.36)

By taking the differential transform on both sides of equation (1.32) we get:

\[ (k+1) \ F(k+1) = \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \ (i!)F(i)(k-i)!F(k-i) \]  \quad (1.37)

Using Eq. (1-34) as recurrence relation and putting coefficient of the \( k=0, 1, \) and 2, in equation (1.34) we can get the coefficient of the power series:

When \( k=0 \) \quad \( F(1) = \left[ F(0) \right]^2 \)

\[ F(1) = 1 \]

When \( k=1 \) \quad \( F(2) = \frac{1}{2} \left[ F(0) \ F(1) + F(1) \ F(0) \right] \)
When \( k = 2 \):

\[
F (2) = 1
\]

\[
F (3) = \frac{1}{3} [F (0) F (2) + F (1) F (1) + F (2) F (0)]
\]

When \( k = 3, 4, \) and \( 5 \) … proceeding in this way, we can find the rest of the coefficients in the power series:

\[
F (r) = F (0) + F (1) + F (2) + F (3) + \ldots = 1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}
\]

**(1.3) Differential Transformation Method for Solving Differential Equations of Lane-Emden Type**

Singular initial value problems in the second order ordinary differential equations occur in several models of mathematical physics and astrophysics [25-27] such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, Isothermal gas spheres and theory of thermo ionic currents which are modelled by means of the following Lane–Emden equation:

\[
u''(x) + \frac{\alpha}{x} u'(x) + f(x, u) = g(x), \quad 0 < x \leq 1, \alpha \geq 0,
\]

Under the following initial conditions

\[
u (0) = A, \ u' (0) = B,
\]

Where \( A \) and \( B \) are constants, \( f(x, u) \) is a continuous real valued function and

\[g (x) \in C[0,1].\]

**Example (1.6):** we first start by considering the following Lane-Emden equation given in:

\[
u' (x) + \frac{2}{x} u(x) + u (x) = 6 + 12x + x^2 + x^3, \quad 0 < x < 1 \quad (1.38)
\]

With initial conditions

\[
u (0) = 0, \ u' (0) = 0. \quad (1.39)
\]

By multiplying both sides of Eq. (1-34) by \( x \) and then taking differential transformation of both sides of the resulting equation using table (1) in sec (1), the following recurrence relation is obtained:

\[
U (k +1) = \frac{1}{(k+1)(k+2)} \times [6\delta (k - 1) + 12\delta(k - 2) + \delta (k - 3) + \delta(k - 4) - \sum_{i=0}^{k} \delta (i - 1)\delta U (k - i)]
\]

\[(1.40)\]
By using equation (1.35) and (1.36), the following transformed initial conditions at \( x_0 = 0 \) can be Obtained:

\[
\begin{align*}
U(0) &= 0, \\
U(1) &= 0,
\end{align*}
\]

(1.41) \hspace{1cm} (1.42)

Substituting equations (1.41) and (1.42) at \( k = 1 \) into equation (1.40), we have:

\[ U(2) = 1, \]  

(1.43)

Following the same recursive procedure, we find \( U(k+1) = 0, \ k = 3, 4, 5 \ldots \) the computation and result corresponding to \( n = 3 \), we have:

\[ U(3) = 1, \]  

(1.44)

Using equations (1.41) - (1.44) and the inverse transformation rule in equation (1.38), we get the solution

\[ U(x) = x^2 + x^3 \]  

(1.45)

**Example (1.7):** We next deem the following Lane-Emden equation given

\[ u''(x) + \frac{9}{x} u' + x u(x) = x^5 - x^4 + 44x^2 - 30x, \quad 0 < x \leq 1 \]  

(1.46)

By using the initial conditions

\[ u(0) = 0, \quad u'(0) = 0, \]  

(1.47)

Then, multiplying both sides of equation (1.44) by \( x \) and then taking differential transformation of both sides of the resulting equation using the table in section 1, we obtain the following recurrence relation:

\[
U(k+1) = \frac{1}{(k+1)(k+2)} \left[ \delta(k-6) - \delta(k-5) + 44\delta(k-3) - 30\delta(k-2) - \sum_{i=1}^{k} \delta(i-2)U(k-i) \right]
\]

(1.48)

We apply the differential transformation at \( x = 0 \), therefore, the initial conditions given in equation (1.47) are transformed as follows:

\[
\begin{align*}
U(0) &= 0, \\
U(1) &= 0,
\end{align*}
\]

(1.49) \hspace{1cm} (1.50)

Substituting equations (1.49) and (1.50) at \( k = 1 \) into equation (1.48), we have

\[ U(2) = 0, \]  

(1.51)

Following the same recursive procedure, we find \( U(k+1) = 0, \ k = 4, 5, 6 \) and listing the computation and result corresponding to \( n = 4 \), we have
Using equations (1.52) - (1.53) and the inverse transformation rule in equation (1.45), we get the following solution:

\[ u(x) = -x^3 + x^4, \]  

(1.54)

For \( n = 4 \), one evaluates that the solution (1.49), which is the solution under the initial conditions in equation (1.47).

(1.4) **General Differential Transformation Method for Higher Order Of Linear Boundary Value Problem**

The differential transformation method (DTM) is one of the numerical methods in ordinary differential, partial differential and integral equations. Since proposed in (Zhou, 1986), there are tremendous interests on the applications of the DTM to solve various scientific problems. For instance, see (Arikoglu & Ozkol, 2005), (Ayaz, 2004), (Bildik et al. 2006), (Chen & Ho, 1999) and (Chen & Liu, 1998). One of the problems that solvable by this method is the boundary value problems (BVPs). This can be observed in (Jang & Chen, 1997), (Erturk & Momani, 2007), (Abdel-Halim Hassan & Erturk, 2009) and (Islam et al., 2009). Previous studies concluded that the DTM can be easily applied in linear and nonlinear differential equations. The DTM is developed based on the Taylor series expansion. This method constructs an analytical solution in the form of polynomial.

**Definition (1.1):**

A Taylor polynomial of degree \( n \) is defined as follows:

\[ f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k \]  

(1.55)

**Theorem (1.1) :**

Suppose that the function \( f \) has \((n+1)\) derivatives on the interval \((c-r, c+r)\), for some \( r > 0 \) and \( \lim_{x \to \infty} R^x x = 0 \), for all \( x \in (c-r, c+r) \) where \( R^n x \) is the error between \( P^n(x) \) and the approximated function \( f(x) \). Then, the Taylor series expanded about \( x = c \) converges to \( x \).

That is,

\[ f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k \]  

(1.56)
For all $x \in (c-r,c+r)$.

**Differential Transformation method:**

Suppose that, the function $f(x)$ is a continuously differentiable function on the interval $(x-r, x+r)$.

**Definition (1.2):**

The differential transform of the function $f(x)$ for the $k$-th derivative is defined as follows:

$$F(x) = \frac{1}{k!} \left[ \frac{d^k f(x)}{dx^k} \right]_{x=x_0}$$

(1.57)

where $f(x)$ is the original function and $F(k)$ is the transformed function.

**Definition (1.3):**

The inverse differential transform of $(k)$ is defined as follows:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} (x-x_0)^k F(k)$$

(1.58)

Substitution of equation (1.57) into equation (1.58) yields:

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)^k \frac{1}{k!} \left[ \frac{d^k f(x)}{dx^k} \right]_{x=x_0}$$

(1.59)

**Proposed Method**

We prove the following theorem by using induction method.

**Theorem (1.2):**

The general differential transformation for BVP of linear differential equation,

$$f(x) = \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_m x^m x \right) e^x$$

is given by

$$F(n+k) = \frac{k!}{(k+n)!} \left[ F(k) + \frac{a_0}{k!} + \sum_{i=1}^{m} a_i \sum_{i=0}^{k} \delta(i-i) \right]$$

(1.60)

for integer $n \geq 1$.

**Proof:**

For $m=1$, we can derive

$$f^{(n)}(x) = f(x) + a_0 e^x + a_1 x e^x$$

(1.61)

The derivative of equation (1.61) for $k \in \mathbb{Z}$ is

$$f^{(n+k)}(x) = f^{(k)}(x) + (a_0 + ka_1) e^x + a_1 x e^x$$

Then,
\[ f^{(n+k)}_{x=0} = f^{(k)}(x) + a_0 + k a_1 \]

By Definition (1.2), we have

\[(n+k)! \ F(n+k) = k! \ F(k) + a_0 + k a_1 \]

\[ F(n+k) = \frac{k![F(k) + \frac{a_0}{k!} + \sum_{i=1}^{m} [a_i \sum_{i=0}^{k} \frac{\delta(i-i)}{(k-1)!}]]}{(n+k)!} \]  \hspace{1cm} (1.62)

For \( m=1 \), equation (1.60) gives

\[ F(n+k) = \frac{k!}{(n+k)!} [F(k) + \frac{a_0}{k!} + \sum_{i=1}^{m} [a_i (\ldots + \frac{1}{(k-1)!} + \ldots)]] \]

\[ F(n+k) = \frac{k![F(k) + \frac{a_0}{k!} + \sum_{i=1}^{m} [a_i (\ldots + \frac{1}{(k-1)!} + \ldots)]]}{(n+k)!} \]  \hspace{1cm} (1.63)

Since equation (1.62) is equal to equation (1.63) Theorem (1.2) holds for \( m=1 \).

Assume that Theorem(1.2) holds for \( m=p \). That is, the differential transformation of

\[ f^{(n)}(x) = f(x) + a_0 e^x + a_1 x e^x + a_2 x^2 e^x + a_3 x^3 e^x + \cdots + a_p x^p e^x \]  \hspace{1cm} (1.64)

Is given by

\[ F(n+k) = \frac{k!}{(n+k)!} [F(k) + \frac{a_0}{k!} + \sum_{i=1}^{m} [a_i (\ldots + \frac{1}{(k-1)!} + \ldots)]] \]  \hspace{1cm} (1.65)

Then for \( m=p+1 \), we have

\[ f^{(n)}(x) = f(x) + [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_p x^p + a_{p+1} x^{p+1}] e^x \]  \hspace{1cm} (1.66)

The derivative of equation (1.66) for \( k \in \mathbb{Z} \) is

\[ f^{(n+k)}(x) \bigg|_{x=0} = f^{(k)}(x) + a_0 + k a_1 + k(k-1)a_2 + k(k-1)(k-2)a_3 + \cdots + k(k-1)(k-2) \cdots (k-p-1)a_p + k(k-1)(k-2) \cdots (k-p)a_{p+1} \]

By Definition (1.2), we have

\[(n+k)! \ F(n+k) = k! \ F(k) + a_0 + k a_1 + k(k-1)a_2 + k(k-1)(k-2)a_3 + \cdots + k(k-1)(k-2) \cdots (k-p-1)a_p + k(k-1)(k-2) \cdots (k-p)a_{p+1} \]
\[
F(n + k) = \frac{k! [F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!} + \cdots + \frac{a_{p+1}}{(k-p-1)!}]}{(n+k)!}
\]

For \(m = p+1\), equation (1-60) gives
\[
F(n + k) = \frac{k! [F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!} + \cdots + \frac{a_{p+1}}{(k-p-1)!}]}{(n+k)!}
\]

From equation (1.65), we have
\[
F(n + k) = \frac{k!}{(n+k)!} [F(k) + \frac{a_0}{k!} + \sum_{i=1}^{p+1} \left( a_i \sum_{j=0}^{n} \frac{\delta(i-j)}{(k-1)!} \right)]
\]

Note that, we have equation (1.67) is equal to equation (1.68). This implies that, Theorem (1.2) holds for \(m=p+1\).

Now, we prove the general form of \(n\)-th order BVPs. For that purpose, we fix \(m=1\).

for \(n=1\), we have
\[
f^{(1)} = f(x) + a_0 e^x + a_1 e^x
\]

The derivative of equation (1.69) for \(k\in\mathbb{Z}\) is
\[
f^{(1+k)} = f^{(k)}(x) + (a_0 + k a_1)e^x + a_1 e^x
\]

Then,
\[
f^{(1+k)}|_{x=0} = f^{(k)}(x) + a_0 + k a_1
\]

By Definition (1.2), we have
\[
(1+k)! \cdot F(1+k) = k! \cdot F(k) + a_0 + k a_1
\]

\[
F(1+k) = \frac{k! [F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!}]}{(1+k)!}
\]

For \(n=1\), equation (1.60) gives
\[
F(1+k) = \frac{k!}{(1+k)!} [F(k) + \frac{a_0}{k!} + a_1]
\]

\[
F(1+k) = \frac{k! [F(k) + \frac{a_0}{k!} + \frac{a_1}{(k-1)!}]}{(1+k)!}
\]
Note that, we have equation (1.70) is equal to equation (1.71). Hence, Theorem (1.2) holds for $n=1$. Assume that, Theorem(1.2) holds for $n=q$. Thus, for $n=q+1$ we have

$$f^{(q+1+k)}|_{x=0} = f^{(k)}(x) + a_0 + k a_1$$

(1.72)

By Definition (1.2), we have

$$(q+1+k)! F(q+1+k) = k! F(k) + a_0 + k a_1$$

$$F(q+1+k) = \frac{k!}{(q+1+k)!} [ F(k) + a_0 + k \frac{a_1}{(k-1)!} ) ]$$

(1.73)

For $n = q+1$, equation (1.60) gives

$$F(q+1+k) = \frac{k!}{(q+1+k)!} [ F(k) + a_0 + k \sum_{i=0}^{k} \frac{(1-i)}{(k-1)i} ]$$

$$F(q+1+k) = \frac{k!}{(q+1+k)!} [ F(k) + a_0 + \frac{a_1}{(k-1)!} ]$$

(1.74)

Note that, we have equation (1.73) is equal to equation (1.74). This implies that, Theorem (1.2) holds for $n=q+1$. 

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Chapter Two

Differential transformation for solving system of differential equations

(2.1): Differential Transform Method for system of Linear Differential Equations

Introduction:
In this paper we apply DTM (one dimensional) on linear differential equation on some example and the result obtained by it are compared with the result obtain by Laplace transform method which are exact solutions. In recent years, Abdel-Halim Hassan I. used differential transform method to solve this type of equations. Arikoglu A applied DTM to obtain numerical solution of differential equations. Ayaz F has used DTM to find the series solution of system of system of differential equations. Bert.W has applied DTM on system of linear equation and analysis its solutions. Chen used DTM to obtain the solutions of nonlinear system of differential equations. Chen C.L. has applied DTM technique for steady nonlinear heat conduction problems. Duan Y used DTM for Burger’s equation to obtain the series solution. Using DTM Hassan have find out series solution and that solution compared with decomposition method for linear and nonlinear initial value problems and prove that DTM is reliable tool to find the numerical solutions. Khaled Batiha has been used DTM to obtain the Taylor’s series as a solution of linear, nonlinear system of ordinary differential equations. Kuo B has been used to find the numerical solution of the solutions of the free convection Problem. Montri Thongmoon has been used to find the numerical solution of ordinary differential equations. The concept of Differential Transform Method (DTM) was first proposed by Zhou and proves that DTM is an iterative procedure for obtaining analytic Taylor’s series solution of differential equations. DTM is very useful to solve equation in ordinary differential equation. It is also applied to solve boundary value problems.

The Differential Transform Method
The transformation of the $k^{th}$ derivative of a function with one variable is follows:

$$U(k) = \frac{1}{k!} \left( \frac{d^k u(x)}{dx^k} \right) \text{ at } x = x_0$$

(2.1)
Where \( u(x) \) is the original function and \( U(k) \) is the transformed function and the differential inverse transformation of \( U(k) \) is defined by,

\[
  u(x) = \sum_{k=0}^{k=x} U(K)(x - x_0)^k
\]

(2.2)

When \( x_0 = 0 \) the function \( u(x) \) defined in (2.2) is express as

\[
  u(x) = \sum_{k=0}^{k=x} U(K)(x)^k
\]

(2.3)

Equation (2.3) implies that the concept of one dimensional differential transform is almost is same as the one dimensional Taylors series expansion. We use following fundamental operation of DTM.

**Numerical Examples**

**Example (2.1):** consider the following system of simultaneous linear differential equations

\[
  \frac{dx}{dt} - 2x + 3y = 0, \quad (2.4)
\]

\[
  \frac{dy}{dt} + 2x - y = 0, \quad (2.5)
\]

With the condition \( x(0) = 8, y(0) = 3 \)

Taking the differential transform method to equation (2.4) and (2.5), using above mentioned theorem we obtain

\[
  (k+1)X(k+1) = 2X(k) - 3Y(k) \quad (2.6)
\]

\[
  (k+1)Y(k+1) = Y(k) - 2X \quad (2.7)
\]

With initial conditions \( X(0) = 8, Y(0) = 3 \),

Take \( k = 0 \), we get \( X(1) = 7 \quad Y(1) = -13 \)

Take \( k = 1 \), we get \( X(2) = \frac{53}{2} \quad Y(2) = -\frac{27}{2} \)

Take \( k = 2 \), we get \( X(3) = \frac{187}{6} \quad Y(3) = -\frac{135}{6} \)

The approximation solution when \( n = 3 \) (number of terms) using equation (2.3) is given by

\[
  x(t) = \sum_{k=0}^{k=3} X(k)t^k
\]

\[
  y(t) = \sum_{k=0}^{k=3} Y(k)t^k
\]
\[
X(t) = 8 + 7t + \frac{53}{2}t^2 + \frac{187}{6}t^3 \\
Y(t) = 3 - 13t + \frac{-27}{2}t^2 + \frac{-135}{6}t^3 
\]

Which the exact solution.

**Example (2.2):** Take the following system of non-homogeneous differential equations:

\[
\begin{align*}
\frac{dx(t)}{dt} &= z(t) - \cos t \\
\frac{dy(t)}{dt} &= z(t) - e^t \\
\frac{dz(t)}{dt} &= x(t) - y(t)
\end{align*}
\]

With the conditions \(x(0) = 1, y(0) = 0, z(0) = 2\)

Taking the DTM to equation (2.8), (2.9) and (2.10) using above mentioned theorem we obtain

\[
X(k + 1) = \frac{1}{k + 1} \left[ Z(k) - \frac{1}{k!} \cos \left( \frac{nk}{2} \right) \right]
\]

\[
Y(k + 1) = \frac{1}{k + 1} \left[ Z(k) - \frac{1}{k!} \right]
\]

\[
Z(k + 1) = \frac{1}{k + 1} \left[ X(k) - Y(k) \right]
\]

With initial conditions \(X(0) = 1, Y(0) = 0, Z(0) = 2\)

Put \(k = 0\) we get \(Z(1) = 1, Y(1) = 1, X(1) = 1\)

Put \(k = 1\) we get \(Z(2) = 0, Y(2) = 0, X(2) = \frac{1}{2}\)

Put \(k = 2\) we get \(Z(3) = \frac{1}{6}, Y(3) = -\frac{1}{6}, X(3) = \frac{1}{6}\)

Put \(k = 3\) we get \(Z(4) = \frac{1}{120}, Y(4) = \frac{1}{120}, X(4) = \frac{1}{24}\)

The approximation solution when \(n = 4\) (number of terms) using equation (2.3) is given by

\[
x(t) = \sum_{k=0}^{k=4} X(k) t^k \\
y(t) = \sum_{k=0}^{k=4} Y(k) t^k
\]
Putting all table values in equation (2.8), (2.9) and (2.10) we get:

\[ x(t) = 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \frac{1}{12} t^4 \]

\[ y(t) = t - \frac{1}{12} t^3 \]

\[ z(t) = 2 + t + \frac{1}{4} t^2 + \frac{1}{6} t^3 + \frac{1}{12} t^4 \]

The exact solution of the problem.

(2.2) Differential Transform Method for Solving Linear and Non-Linear System of Ordinary Differential Equations

Know, differential transform method (DTM) is applied to linear system and non-linear system of ODE.

Example (2.3): consider the following non-linear system of differential equations

\[
\begin{align*}
\dot{y}_1(x) &= 2e^{4x}y_4^2(x) \\
\dot{y}_2(x) &= y_1(x) - y_3(x) + \cos(x) - e^{2x} \\
\dot{y}_3(x) &= y_2(x) - y_4(x) + e^{-x} - \sin(x) \\
\dot{y}_4(x) &= e^{-5x}y_1^2(x)
\end{align*}
\]

(2.11)

With the conditions

\[
\begin{align*}
y_1(0) &= 1 \\
y_2(0) &= 1 \\
y_3(0) &= 0 \\
y_4(0) &= 1
\end{align*}
\]

(2.12)

By using table (1.1.1) in chapter 1, (2.11) and (2.12) transformed as follows:
\[
\begin{align*}
(k + 1)Y_1(k + 1) &= 2 \sum_{k_2 = 0}^{k} \sum_{k_1 = 0}^{k_2} Y_4(k_2 - k_1)Y_4(k - k_2) \frac{4^{k_1}}{k_1!} \\
(k + 1)Y_2(k + 1) &= y_1(k) - Y_3(k) + \frac{1}{k!} \cos \frac{k\pi}{2} - \frac{2^k}{k!} \\
(k + 1)Y_3(k + 1) &= y_2(k) - Y_4(k) + \frac{(-1)^k}{k!} - \frac{1}{k!} \sin \frac{k\pi}{2} \\
(k + 1)Y_4(k + 1) &= -\sum_{k_2 = 0}^{k} \sum_{k_1 = 0}^{k_2} \frac{(-5)^{k_1}}{k_1!} Y_1(k_2 - k_1)Y_1(k - k_2) \\
Y_1(0) &= 1 \quad Y_2(0) = 1 \quad Y_3(0) = 0 \quad Y_4(0) = 1
\end{align*}
\]

Consequently we find
\[
\begin{align*}
Y_1(1) &= 2 \quad Y_2(1) = 1 \quad Y_3(0) = 1 \quad Y_4(0) = 1 \\
Y_1(2) &= 2 \quad Y_2(2) = -\frac{1}{2!} \quad Y_3(2) = 0 \quad Y_4(2) = -\frac{1}{2!} \\
Y_1(3) &= \frac{8}{3!} \quad Y_2(3) = -\frac{1}{3!} \quad Y_3(3) = -\frac{1}{3!} \quad Y_4(3) = -\frac{1}{3!} \\
Y_1(4) &= \frac{16}{4!} \quad Y_2(4) = \frac{1}{4!} \quad Y_3(4) = 0 \quad Y_4(4) = -\frac{1}{4!} \\
Y_1(5) &= \frac{32}{5!} \quad Y_2(5) = -\frac{1}{5!} \quad Y_3(5) = \frac{1}{5!} \quad Y_4(5) = -\frac{1}{5!}
\end{align*}
\]

Therefore, from \(y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k\) the solution is given by

\[
y_1(x) = 1 + 2x + 2x^2 + \frac{8}{3!} x^3 + \ldots = e^{2x}
\]

\[
y_2(x) = \left[x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \ldots \right] + \left[1 - \frac{1}{2!} x^2 + \frac{1}{2!} x^{21} + \ldots \right]
\]

\[
= \sin(x) + \cos(x)
\]

\[
y_3(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \ldots = \sin(x),
\]

\[
y_4(x) = 1 - x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \ldots = e^{-x}
\]

**Example (2.4):** Consider the following system of differential equation of order two

\[
\begin{align*}
(y''_1(x) + y_1(x) + y''_2(x) - 4y_2(x) &= 0 \\
y'(x) + y'_2(x) &= \cos(x) + 2\cos(2x)
\end{align*}
\]

(2.13)

With the conditions:
The transform of equation (2.13) above is begin
\[
\begin{cases}
(k + 2)(k + 1)Y_1(k + 2) + Y_1(k) - (k + 2)(k + 1)Y_2(k + 2) - 4Y_2(k) = 0 \\
(k + 1)Y_1(k + 1) + (k + 1)Y_2(k + 1) = \frac{1}{k!} \cos\left(\frac{k\pi}{2}\right) + \frac{2^{k+1}}{k!} \cos\left(\frac{k\pi}{2}\right)
\end{cases}
\]
Y_1(0) = 0, Y_1(1) = 1
Y_2(0) = 0, Y_2(1) = 2

Consequently, we find
\[
Y_1(2) = 0, \quad Y_2(2) = 0
\]
\[
Y_1(3) = 0, \quad Y_2(3) = -\frac{8}{3!}
\]
\[
Y_1(4) = 0, \quad Y_2(4) = 0
\]
\[
Y_1(5) = 0, \quad Y_2(5) = \frac{32}{5!}
\]

Therefore from above the solution of (2.13) is given by:
\[
y_1(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sin(x)
\]
\[
y_2(x) = 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 \cdots = \sin(2x)
\]

(2.3) Exact Solutions for A Class of Stiff System By Differential Transform Method

Consider the stiff initial value problem
\[
y'(x) = f\left(x, y(x)\right), \quad y(x_0) = y_0
\]

On the finite interval \(I = (x_0, x_N)\) where
\(Y : (x_0, x_n) \rightarrow \mathbb{R}^m\) and \(f : (x_0, x_n) \times \mathbb{R}^m \rightarrow \mathbb{R}^m\) are continuous.

**Definition:** if the solution of the system contains components which changes at significantly different rates for given changes in the independent variable, then system is said to be stiff in this section, we apply DTM to both linear and nonlinear stiff system.

Stiff differential equations are characterized as those whose exact solution has a term
of the form $e^{-\delta t}$.

**Example (2.5):** consider the linear stiff system:

\[ y_1' = -y_1 - 15y_2 + 15e^{-x} \quad (2.14) \]
\[ y_2' = 15y_1 - y_2 - 15 \quad (2.15) \]

with initial value $y_1(0) = 1, y_2(0) = 1$

This system has eigen value of large modulus lying close to the imaginary axis

$-1 \pm 15i$

By applying differential transformation, we have

\[ y_1(k + 1) = \frac{1}{(k+1)} \left[ -y_1(k) - 15y_2(k) + \frac{15(-1)^k}{ki} \right] \quad (2.16) \]
\[ y_2(k + 1) = \frac{1}{(k+1)} \left[ 15y_1(k) - y_2(k) - \frac{15(-1)^k}{ki} \right] \quad (2.17) \]

The initial conditions of DTM are given

$Y_1(0) = 0, \quad Y_2(0) = 1$

For $k = 0, 1, 2, 3\ldots$ the series coefficients for $y_1(k)$ and $y_2(k)$ can be obtained as

\[ y_1(0) = 1, \quad y_1(1) = -1, \quad y_1(2) = \frac{1}{2i}, \quad y_1(3) = -\frac{1}{3i} \]
\[ y_1(4) = \frac{1}{4i}, \quad y_1(5) = -\frac{1}{5i} \]
\[ y_2(0) = 1, \quad y_2(1) = -1, \quad y_2(2) = \frac{1}{2i}, \quad y_2(3) = -\frac{1}{3i} \]
\[ y_2(4) = \frac{1}{4i}, \quad y_2(5) = -\frac{1}{5i} \]

Using the inverse transform method we get:

\[ y(x) = \sum_{k=0}^{\infty} x^k y(k) \quad (2.18) \]
\[ y_1(x) = 1 - \frac{x}{1i} + \frac{x^2}{2i} - \frac{x^3}{3i} + \frac{x^4}{4i} + \ldots = e^{-x} \quad (2.19) \]
\[ y_2(x) = 1 - \frac{x}{1i} + \frac{x^2}{2i} - \frac{x^3}{3i} + \frac{x^4}{4i} + \ldots = e^{-x} \quad (2.20) \]
**Example (2.6):** consider the non-linear system in the form of initial value example (2.5) is given by:

\[ y'_1 = -1002y_1 + 1000y_2^2, \quad y_1(0) = 1 \]  \tag{2.21}

\[ y'_2 = y_1 - y_2 - y_2^2, \quad y_2(0) = 1 \]  \tag{2.22}

\[
Y_1(k + 1) = \frac{1}{(k+1)} \left[-1002Y_1(k) + 1000 \sum_{r=0}^{k} Y_1(r)Y_1(k-r)\right] \tag{2.23}
\]

\[
Y_2(k + 1) = \frac{1}{(k+1)} \left[Y_1(k) - Y_2(k) - \sum_{r=0}^{k} Y_1(r)Y_2(k-r)\right] \tag{2.24}
\]

For \(k=0, 1, 2, 3\ldots \text{n}\) the series coefficients for \(Y_1(k)\) and \(y_2(k)\) can be obtained as

\[ Y_1(0) = 1, \; Y_1(1) = -2, \; Y_1(2) = 2, \; Y_1(3) = -\frac{4}{3} \]

\[ Y_1(4) = -\frac{2}{3}, \; Y_1(5) = 4 \]

\[ Y_2(0) = 1, \; Y_2(1) = -1, \; Y_2(2) = \frac{1}{2}, \; Y_2(3) = -\frac{1}{3!} \]

\[ Y_1(4) = -\frac{1}{4!}, \; Y_2(5) = -\frac{1}{5!} \]

Using the inverse transform:

\[ y_1(x) = 1 - \frac{2x}{1!} + \frac{4x^2}{2!} + \frac{2x^3}{3!} + \frac{16x^4}{4!} + \cdots = e^{-2x} \]  \tag{2.25}

\[ y_2(x) = -\frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots = e^{-x} \]  \tag{2.26}

The equations (2.25) and (2.26) are the exact solution by differential transform method.
Chapter Three

Solution of complex differential equation system by using differential transform method

There, using differential transform method second order complex different equation system was solved firstly we separated real and imagine parts these equations system .thus from two unknown equation system four equality was obtained . Later using two dimensional differential transform we obtained real and imagined parts of solutions.

In complex partial differential equation was solved. But the concept of differential transform (one- dimension) was first proposed and applied to solve linear and nonlinear initial value problems in electric circuit analysis by Zhou (28): solving partial differential equations by two dimensional differential transform method was proposed by chao kuang cheu and shing huei ho by (29): partial differential equation was solved by using two dimensional DTM in (29)-(30). System of differential equation was solved using two dimensional DTM in (31).

Let w=w (Z, Ź) be a complex function

here Z=x+iy , w(Z , Ź )= U(X , Y) + Iv(x , y) . Derivative according to Z and Ź of w (Z, Ź) is defined as follows:

\[
\frac{dw}{dz} = \frac{1}{2} \left( \frac{dw}{dz} - i \frac{dw}{dz} \right) \\
\frac{dw}{d\bar{z}} = \frac{1}{2} \left( \frac{dw}{dz} + i \frac{dw}{dz} \right)
\]

Here

\[
\frac{dw}{dz} = \left( \frac{du}{dx} - i \frac{dv}{dy} \right) \\
\frac{dw}{d\bar{z}} = \left( \frac{du}{dx} + i \frac{dv}{dy} \right)
\]

Similarly second order derivatives according to Z and Ź are defined as follows:

\[
\frac{d^2w}{dz^2} = \frac{1}{4} \left( \frac{d^2w}{dx^2} - 2i \frac{d^2w}{dxdy} - \frac{d^2w}{dy^2} \right)
\]

\[
\frac{d^2w}{d\bar{z}^2} = \frac{1}{4} \left( \frac{d^2w}{dx^2} + 2i \frac{d^2w}{dxdy} - \frac{d^2w}{dy^2} \right)
\]
Two Dimensional Differential Transform

**Definition 3.1:** two dimensional differential transform of function, \( f(x, y) \) is defined as follows

\[
f(k, h) = \frac{1}{k!h!} \left[ \frac{d^{k+h} f(x, y)}{dx^k dy^h} \right]_{x=0, y=0}
\]  

(3.8)

In equation (8) \( f(x, y) \) is original function and \( f(k, h) \) is transformed function, which is called T-function.

**Definition 3.2:** differential inverse transform of \((k, h)\) is defined as follows:

\[
\sum_{k=0}^{\infty} \sum_{h=0}^{\infty} f(k, h) x^k y^h
\]

Equation dimples that the concept of two dimensional differential transform is derived from two dimensional Taylor series expansion.

- **Theorem 3(2):** if \( w(x, y) = u(x, y) \pm v(x, y) \) then \( w(k, h) = u(k, h) \pm v(k, h) \)
- **Theorem 4(2):** if \( w(x, y) = \lambda u(x, y) \) then \( w(k, h) = \lambda u(k, h) \pm v(k, h) \)
- **Theorem 5(2):** if \( w(x, y) = \frac{du(x,y)}{dx} \) then \( w(k, h) = (k+1) u(k+1, h) \)
- **Theorem 6(2):** if \( w(x, y) = \frac{du(x,y)}{dy} \) then \( w(k, h) = (h+1) u(k, h+1) \)
- **Theorem 7(2):** if \( w(x, y) = \frac{d^{r+s} f(x,y)}{dx^r dy^s} \) then \( w(k, h) = (k+1)(h+2)\ldots(k+r)(h+1)(h+2)\ldots(h+s) u(k+r, h+s) \)
- **Theorem 8(2):** if \( w(x, y) = u(x, y) \cdot v(x, y) \) then \( w(k, h) = \sum_{p=0}^{k} \sum_{s=0}^{h} u(r, h-s) v(k-r, s) \)
- **Theorem 9(2):** if \( w(x, y) = x^m y^n \) then \( w(k, h) = \delta(k-m) \delta(h-n) \)

**Example (3-1):** Solve the following complex differential equation system.

\[
\frac{dw_1}{dz} + \frac{dw_2}{\partial Z} = 2z + 3 \tag{3.10}
\]

\[
\frac{dw_1}{dz} + \frac{dw_2}{\partial Z} = Z
\]
With initial condition

\[ W_1(x, 0) = x^2 + 2x \]  \hspace{1cm} (3.11)
\[ W_2(x, 0) = 8x \]  \hspace{1cm} (3.12)

\[ w_1 = u_1 + i v_1 \quad , \quad w_2 = u_2 + i v_2 \]

and from (3.1), (3.2) system (3.10)-(3.11) I equivalent that equation (3.10) become

\[
\frac{1}{2} \left( \frac{d w_1}{d x} + i \frac{d w_1}{d y} \right) + \frac{1}{2} \left( \frac{d w_2}{d x} - i \frac{d w_2}{d y} \right)
\]
\[
\frac{1}{2} \left( \frac{d u_1}{d x} + i \frac{d u_1}{d y} - \frac{d v_1}{d x} + i \frac{d v_1}{d y} + \frac{d u_2}{d x} + i \frac{d u_2}{d y} - \frac{d v_2}{d x} - i \frac{d v_2}{d y} \right) = 2x + 2y + 3
\]

\[
\Rightarrow \frac{d u_1}{d x} - \frac{d v_1}{d y} + \frac{d u_2}{d x} - \frac{d v_2}{d y} = 4x + 6 \tag{3.13}
\]

\[
\frac{d v_1}{d x} - \frac{d u_1}{d y} + \frac{d v_2}{d x} = 4y \tag{3.14}
\]

From equation (3.11) we get:

\[
\frac{1}{2} \left( \frac{d w_1}{d x} + i \frac{d w_1}{d y} \right) + \frac{1}{2} \left( \frac{d w_2}{d x} - i \frac{d w_2}{d y} \right) = 7
\]

\[
\Rightarrow \frac{1}{2} \left( \frac{d u_1}{d x} + i \frac{d u_1}{d y} - \frac{d v_1}{d x} + i \frac{d v_1}{d y} + \frac{d u_2}{d x} + i \frac{d u_2}{d y} - \frac{d v_2}{d x} - i \frac{d v_2}{d y} \right) = 7
\]

\[
\Rightarrow \frac{d u_1}{d x} - \frac{d v_1}{d y} + \frac{d u_2}{d x} + \frac{d v_2}{d y} = 2 \tag{3.15}
\]

\[
\frac{d v_1}{d x} + \frac{d u_1}{d y} + \frac{d v_2}{d x} = 0 \tag{3.16}
\]

From above and (3.11), (3.12) initial condition we have that:

\[
U_1 (0, 0) = 0 \quad u_1 (1, 0) = 2 \quad u_1 (2, 0) = 1 \quad u_1 (i, 0) = 0 \quad i > 2
\]

\[
U_1 (i, 0) = 0 \quad (i \in \mathbb{N})
\]

\[
U_2 (i, 0) = 8 \quad u_2 (i, 0) = 0 \quad (i \geq 2) \quad u_2 (i, 0) = 0
\]

\[
U_2 (0, 0) = 0 \quad (i \in \mathbb{N})
\]

From differential transform of (3.13), (3.14),(3.15) and (3.16) is get that

\[
(k+1) u_1 (k+1, h) + (h+1) u_1 (k, h+1) + (k+1) u_2 (k+1, h) - (h+1) u_2 (k, h+1) = 48 (k-1, h) + 68 (k, h) \tag{3.17}
\]

\[
(k+1) u_1 (k+1, h) - (h+1) u_1 (k, h+1) + (h+1) (k+1) u_2 (k, h+1) + (k+1) u_2 (k+1, h) = 48 (k, h-1) \tag{3.18}
\]
(k+1) u_1 (k+1, h) - (h+1) u_1 (k, h+1) + (k+1) u_2 (k+1, h) +
(h+1) u_2 (k, h+1) = k_1 \delta (k+h)
(3.19)

(k+1) u_1 (k+1, h) + (h+1) u_1 (k, h+1) + (k+1) u_2 (k+1, h) - (h+1) u_2 (k, h+1) = 0
(3.20)

U_1(0,1) - u_2(0,1) = -4, u_1(0,1) = a, u_2(0,1) = a+4

a \in \mathbb{R}
(3.21)

By sum of (3.17) and with (3.19)

U_1 (k+1, h) + u_2 (k+1, 1) =

\frac{48 (k,h-1)+208 (k,h)}{2 (k+1)}
(3.22)

By mines of (3.18) and (3.19) we get that

U_2 (k, h+1) + u_1 (k, h+1) = \frac{48 (k,h-1)}{2 (h+1)}
(3.23)

By mines of (3.17) with (3.18) we get that

u_1 (k, h+1) - u_2 (k, h+1) = \frac{48 (k,h+1)-88 (k,h)}{2 (h+1)}
(3.24)

By sum of (3.19) with (3.20) we get that:

v_1 (k+1, h) + v_2 (k+1, h) = \frac{48 (h-1)}{2 (k+1)}
(3.24)

If we write in place of h, h + 1 in (3.22) we get that

U_1 (k+1, h+1) + u_2 (k+1, h+1) =

\frac{48 (k,h)+208 (h+1)}{2 (k+1)}
(3.25)

If we write in place of k, k + 1 in (3.23) we get that

U_2 (k+1, h+1) = \frac{48 (k,h)-88 (k+1,h)}{2 (h+1)}
(3.26)

If we write in place of k, h +1 in (3.24) we get that

v_1 (k+1, h+1) - v_1 (k+1, h+1) =

\frac{48 (k,h)-88 (k+1,h)}{2 (h+1)}
(3.27)

If we write in place of h, h +1 in (3.25) we get that
\[ V_1 (k+1, h+1) + v_2 (k+1, h+1) = \frac{48 (k, h)}{2 (k+1)} \] \hspace{1cm} (3.28)

By sum of (3.26) with (3.27) we get that:

\[ U_2 (k+1, h+1) = \]

\[ \frac{8 (k, h) - 58 (k, h+1)}{k+1} + \frac{8 (k+1, h-1)}{k+1} \] \hspace{1cm} (3.29)

By mines of (3.27) from (3.26) we get that:

\[ U_1 (k+1, h+1) = \]

\[ \frac{8 (k, h) + 58 (k, h+1)}{k+1} - \frac{8 (k+1, h-1)}{k+1} \] \hspace{1cm} (3.30)

By sum of (3.28) with (3.29) we get that:

\[ V_1 (k+1, h+1) = \]

\[ \frac{8 (k, h) - 28 (k+1, h)}{k+1} + \frac{8 (k, h)}{k+1} \] \hspace{1cm} (3.31)

By mines of (3.28) with (3.29) we get that:

\[ v_2 (k+1, h+1) = \]

\[ \frac{8 (k, h) - 28 (k+1, h)}{k+1} - \frac{8 (k, h)}{k+1} \] \hspace{1cm} (3.32)

In (3.32) equality if we write \( k = 0, h = 0 \) than is obtained

\[ V_1 (1, 1) = 2 \] \hspace{1cm} (3.33)

In (3.18) equality if we write \( k = 0, h = 1 \) than by using (3.29) is obtained

\[ U_2(0,2) + U_1(0,2) = 1, U_1(0,2) = b, U_2(0,2) = b, b \in \mathbb{R} \] \hspace{1cm} (3.34)

If we write \( k = 0, h = 0 \) in (3.23) we get that

\[ U_1(0,1) = U_2(0,1) = c, c \in \mathbb{R} \] \hspace{1cm} (3.35)

By using (3.30), (3.31), (3.32) and (3.33) it is seen that all other components of \( u_1, v_1, u_2 \) and \( v_2 \) is zero, thus it is obtained:

\[ u_1(x, y) = \sum_{k=0}^{\infty} \left( \sum_{h=0}^{\infty} U_1 (k, h)x^k y^h \right) \]

\[ = x^2 + by^2 + 2x + cy \] \hspace{1cm} (3.36)
\[ v_1(x, y) = \sum_{k=0}^{\infty} \left( \sum_{h=0}^{\infty} V_1(k, h)x^ky^h \right) \]
\[ = 2x^2y + ay \quad \text{(3.37)} \]
\[ u_2(x, y) = \sum_{k=0}^{\infty} \left( \sum_{h=0}^{\infty} U_2(k, h)x^ky^h \right) \]
\[ = 8x + (b + 1)y^2 + cy \quad \text{(3.38)} \]
\[ v_2(x, y) = \sum_{k=0}^{\infty} \left( \sum_{h=0}^{\infty} V_2(k, h)x^ky^h \right) \]
\[ = (a + 4)y \quad \text{(3.39)} \]

From (3.38) - (3.39) we get:

\[ W_1(x, y) = u_1(x, y) + v_1(x, y) \]
\[ = x^2 + by^2 + 2x + cy + i(2x - y + ay) \]
\[ W_2(x, y) = u_2(x, y) + v_2(x, y) \]

**Example (3.2):** solve the following complex differential equation system

\[ 2 \frac{dw_1}{dz} \frac{dw_2}{dz} = 6z^2 - 4z \quad \text{(3.40)} \]
\[ \frac{dw_1}{dz} - 3 \frac{dw_2}{dz} = 12\hat{Z} - 2 \quad \text{(3.41)} \]

With initial conditions:

\[ W_1(x, 0) = x^3 - 2x \quad \text{(3.42)} \]
\[ W_2(x, 0) = 4x^2 \quad \text{(3.43)} \]

If we write system in (3.40), (3.41) system \( w_1 = u_1 + iu_1, w_2 = u_2 + iu_2 \)

We get the following equation:

\[ 2(\frac{du_1}{dx} + i\frac{dv_1}{dx} - \frac{du_1}{dy} - i\frac{dv_1}{dy}) - (\frac{du_2}{dx} + i\frac{dv_2}{dx} + i\frac{du_2}{dy} - \frac{dv_2}{dy}) = 2(x + iy) - 8(x + iy) \]
\[ \frac{du_1}{dx} - \frac{dv_1}{dy} + i\frac{du_1}{dx} + i\frac{dv_1}{dx} + 3\frac{du_1}{dx} + i\frac{dv_1}{dx} + 3\frac{du_1}{dx} + 3\frac{dv_1}{dx} \]
\[= 24(x - iy) - 4 \quad (3.44)\]

If (3.44) equities is separated into read and imaginary parts then it is get following equalities.

\[
\frac{2du_1}{dx} + 2\frac{dv_1}{dy} - \frac{du_2}{dx} + \frac{dv_2}{dy} = 12(x^2 - y^2) - 8x \quad (3.45)
\]

\[
2\frac{dv_1}{dx} - 2\frac{dv_1}{dy} - \frac{dv_2}{dx} - \frac{du_2}{dy} = 24xy - 8y \quad (3.46)
\]

\[
\frac{du_1}{dx} - \frac{dv_1}{dy} + 3\frac{dv_2}{dx} + 3\frac{dv_2}{dy} = 24x - 4 \quad (3.47)
\]

\[
\frac{dv_2}{dx} + 3\frac{dv_2}{dx} - 3\frac{du_2}{dy} = -24y \quad (3.48)
\]

From (3.42) - (3.43) initial conditions we have that:

\[
U_1 (0,0) = 0 \quad U_1 (1,0) = -2 \quad U_1 (2,0) = 0 \quad U_1 (3,0) = 1
\]

\[
U_1 (i,0) = 0 \quad (i > 3), \quad V_1 (i,0) = 0 \quad (i \in \mathbb{N}) \quad (3.49)
\]

From differential transform of (3.45) – (3.48) get following equalities:

\[
2(k+1) u_1 (k+1,h) + 2(h+1) v_1 (k,h+1) - (k+1)u_2 (k+1,h) + (h+1)v_2 (k,h+1) = 128(k-2,h) - 12(k,h-2) - 88(k-1,h) \quad (3.50)
\]

\[
2(k+1) v_1 (k+1,h) - 2(h+1) u_1 (k,h+1) - (k+1)v_2 (k+1,h) + (h+1)u_2 (k,h+1) = 248(k-1,h-1) - 8(k,h-2) \quad (3.51)
\]

\[
(k+1) u_1 (k+1,h) - (h+1) v_1 (k,h+1) - 3(k+1)
\]

\[
u_2 (k+1,h) + 3(h+1) v_2 (k,h+1) = 248(k-1,h) - 48(k,h) \quad (3.52)
\]

\[
(k+1) u_1 (k,h+1) + (k+1) v_1 (k+1,h) + 3(k+1)v_2 (k+1,h) - 3(h+1)u_2 (k,h+1) = -248(k,h-1) - 12(k,h-2) - 8(k-1,h) \quad (3.53)
\]

If we write \( h = 0 \) in (3.51) and (3.52) from (3.49) we get that:

\[
U_1 (k,1) = u_2 (k,1) = 0 \quad (3.54)
\]

If write \( k=0, h=0 \) in (3.41) and (3.52) from (3.49) we get that:

\[
v_1 (0,1) = 2, \quad v_2 (0,1) = 0 \quad (3.55)
\]

If write \( k=0 \) in (3.51) and (3.53) from (3.49) we get that:

\[
u_1 (k,1) = 2, \quad u_2 (k,1) = 0 \quad (3.56)
\]
If write $k=1$, $h=0$ in (3.50) and (3.52) from (3.49) we get that:

$$v_1(1, 1) = 0, \quad v_2(1, 1) = 0$$  \hfill (3.57)

If write $k=0$, $h=0$ in (3.51) and (3.53) from (3.49) we get that:

$$u_1(0, 2) = 0, \quad u_2(0, 2) = 4$$  \hfill (3.58)

If write $h=1$ in (3.50) and (3.52) from (3.48) and (3.54) we get that:

$$v_1(k, 2) = 0, \quad v_2(k, 2) = 0$$  \hfill (3.59)

If write $h=2$ in (3.51) and (3.53) from (3.59) we get that:

$$u_1(k, 3) = u_2(k, 3) = 0$$  \hfill (3.60)

If write $k=2$, $h=0$ in (3.50) and (3.52) we get that:

$$v_1(2, 1) = 3, \quad v_2(2, 1) = 0$$  \hfill (3.61)

If write $k=1$, $h=1$ in (3.51) and (3.53) we get that:

$$u_1(1, 2) = -3, \quad u_2(1, 2) = 0$$  \hfill (3.62)

By using (3.53) – (3.61) we see other components of $u_1$, $u_2$, $v_1$ and $v_2$ are equal zero thus it is obtained

$$U_1(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} u_1(k, h)x^ky^k = x^3 - 3xy^2 - 2x$$

$$U_2(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} u_2(k, h)x^ky^k = 3xy^2 - y^2 + 2y$$

$$V_1(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} v_1(k, h)x^ky^k = 4x^2 + 4y^2$$

$$V_2(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} v_2(k, h)x^ky^k = 0$$

$$w_1(x, y) = u_1(x, y) + v_1(x, y) = x^3 - 3xy^2 - 2x + i(3xy^2 - y^3 + 2y) = Z^3 - 2\hat{Z}$$

$$w_2(x, y) = u_2(x, y) + v_2(x, y) = 4x^2 + 4y^2 = 4Z\hat{Z}$$
Chapter Four

The applications of DTM

(4-1): Differential Transformation Method for a Reliable Treatment of the Nonlinear Biochemical Reaction Model

1 Introduction

The concept of differential transformation method was first proposed by Zhou [32] in 1986 (see [33, 34]), and it was applied to solve linear and non-linear initial value problems in electric circuit analysis. This method constructs a semi analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The Differential transformation method is very effective and powerful for solving various kinds of differential equations. For example, it was applied to two point boundary value problems [35, 1986], to differential-algebraic equations [36], to the KdV and mKdV equations [37], to the Schrodinger equations [38], to fractional differential equations [39] and to the Riccati differential equation [40]. Jang et al. [41] introduced the application of the concept of the differential transformation of fixed grid size to approximate solutions of linear and non-linear initial value problems. Hassan [42] applied the differential transformation technique of fixed grid size to solve the higher-order initial value problems. The transformation method can be used to evaluate the approximating solution by the finite Taylor series and by an iteration procedure described by the transformed equations obtained from the original equation using the operations of differential transformation.

\[ E + A \leftrightarrow Y \rightarrow E + X, \quad (4.1) \]

Where \( E \) is the enzyme, \( A \) the substrate, \( Y \) the intermediate complex and \( X \) the product. The time evolution of scheme (1) can be determined from the solution of the system of coupled nonlinear ODEs

\[ \frac{dA}{dt} = -k_1 EA + k_{-1} Y, \quad (4.2) \]
\[
\begin{align*}
\frac{dE}{dt} &= -k_1EA + (k_{-1} + k_2)Y, \quad (4.3) \\
\frac{dY}{dt} &= k_1EA - (k_{-1} + k_2)Y, \quad (4.4) \\
\frac{dX}{dt} &= k_2Y, \quad (4.5)
\end{align*}
\]

Subject to the initial condition
\[A(0) = A_0, \quad E(0) = E_0, \quad Y(0) = Y_0, \quad X(0) = X_0 \quad (4.6),\]

Where the parameters \(k_1, k_{-1}\) and \(K_2\) are positive rate constants for each reaction. Systems (4.2)–(4.5) can be reduced to only two equations for \(A\) and \(Y\)

and in dimensionless form of concentrations of substrate, \(x\), and intermediate complex between enzyme and substrate, \(y\), are given by [13]
\[
\begin{align*}
\frac{dx}{dt} &= -x + (\beta - \alpha)y + xy, \quad (4.7) \\
\frac{dy}{dt} &= \frac{1}{\epsilon}(x - \beta y - xy), \quad (4.8)
\end{align*}
\]

subject to initial conditions
\[
x(0) = 1, \quad y(0) = 0, \quad (4.9)
\]

where \(\alpha, \beta\) and \(\_\) are dimensionless parameters.

There, we present a reliable algorithm based on DTM to find solution of the system of coupled nonlinear ODEs (4.7) and (4.8).

**Application**

By using the fundamental operations of differential transformation method in Table (1.1.1) chapter 1, we obtained the following recurrence relation to the system (4.7) and (4.8);

\[
X(k + 1) = \frac{1}{(k + 1)} \left[ -X(k) + \beta Y(k) - \alpha Y(k) \\
+ \sum_{m=0}^{k} X(m)Y(k - m) \right] \quad (4.10)
\]

\[
Y(k + 1) = \frac{1}{(k + 1)} \left[ \frac{1}{\epsilon} X(k) - \frac{\beta}{\epsilon} Y(k) \\
- \frac{1}{\epsilon} \sum_{m=0}^{k} X(m)Y(k - m) \right] \quad (4.11)
\]

From the initial condition \(x(0) = 1, \ y(0) = 0\), we have \(X(0) = 1, Y(0) = 0\), and from equations (4.10) and (4.11) and for the case \(\beta = 1.0, \ \alpha = 0.375,\) and \(\epsilon = 0.1\) we get
\[ X(0) = 1, \ X(1) = -1, \ X(2) = \frac{69}{8}, \ X(3) = -\frac{757}{12}, X(4) = \frac{47767}{128}, \]

\[
\ldots \tag{4.12}
\]

\[ Y(0) = 0, Y(1) = 10, Y(2) = -105, Y(3) = \frac{9145}{12}, \\
Y(4) = \frac{-17785}{4}, \ldots \tag{4.13}
\]

Therefore,

\[
x(t) = \sum_{k=0}^{\infty} X(k) t^k = 1 - t + \frac{69}{8} t^2 - \frac{757}{12} t^3 + \frac{47767}{128} t^4 - \frac{3800401}{1920} t^5 \\
+ \ldots \tag{4.14}
\]

\[
y(t) = \sum_{k=0}^{\infty} Y(k) t^k = 10 t - 105 t^2 + \frac{9145}{12} t^3 - \frac{17785}{4} t^4 + \frac{4440661}{192} t^5 + \ldots \tag{4.15}
\]

**Applications of the two-dimensional differential transform for solving nonlinear wave equations**

**Introduction**

Recently, analytical method is widely used for solving linear and nonlinear equations. The Differential Transformation Method (DTM) is one of the analytical methods employed in this article which was firstly applied in the engineering gamut by Zhou in 1986 for solving linear and nonlinear equation [43]. This method achieves a solution based on Taylor series. The differential transform method has been developed for solving the differential equations. For example, in [44-46] this method applied to partial differential equations.

**Method of solution**

**Two- dimensional differential transformation method**

In this section, the fundamental idea of two-dimensional differential transform method (2D DTM) is concisely introduced [22-25]. We assume a function \( u(t, x) \) that is analytic and differentiated continuously with respect to time \( t \) in the domain of interest, then let

\[
w(k, h) = \frac{1}{k! h!} \left[ \frac{\partial^{k+h} w(t, x)}{\partial t^k \partial x^h} \right]_{(t, x) = (0, 0)} \tag{4.16}
\]

Where \( w(t, x) \) is the origin function and \( W(k, h) \) is transform function. The transformation is called the T- function. The differential inverse transform of \( W(k, h) \) is defined as follows
\[ w(t,x) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k,h)t^k x^h \]  

(4.17)

From equation (4.16) and equation (4.17) can be concluded

\[ w(t,x) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k! h!} \left[ \frac{\partial^{k+h} w(t,x)}{\partial t^k \partial x^h} \right]_{(t,x)=(0,0)} t^k x^h \]  

(4.18)

And equation (4.17) can be rewritten as

\[ w(t,x) = \sum_{k=0}^{m} \sum_{h=0}^{n} W(k,h)t^k x^h \]  

(4.19)

Equation (4.19) implies that

\[ w(t,x) = \sum_{k=m+1}^{\infty} \sum_{h=n+1}^{\infty} W(k,h)t^k x^h \]  

(4.20)

Some of the fundamental functions and transformed functions are listed in Error! Reference source not found...

<table>
<thead>
<tr>
<th><strong>Origin function</strong></th>
<th><strong>Transformed function</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( w(t,x) = u(t,x) \pm v(t,x) )</td>
<td>( W(k,h) = U(k,h) \pm V(k,h) )</td>
</tr>
<tr>
<td>( w(t,x) = \frac{\partial w(t,x)}{\partial t} )</td>
<td>( W(k,h) = (k+1)W(k+1,h) )</td>
</tr>
<tr>
<td>( w(t,x) = \frac{\partial w(t,x)}{\partial x} )</td>
<td>( W(k,h) = (h+1)W(k,h+1) )</td>
</tr>
<tr>
<td>( w(t,x) = \frac{\partial^{r+q} w(t,x)}{\partial t^r \partial x^q} )</td>
<td>( W(k,h) = (k+1)(k+2) \ldots (k+r)(h+1)(h+2) \ldots (h+q) )</td>
</tr>
<tr>
<td>( w(t,x) = u(t,x)v(t,x) )</td>
<td>( W(k,h) = \sum_{i=0}^{k} \sum_{j=0}^{h} U(i,h-j)V(k-i,j) )</td>
</tr>
<tr>
<td>( w(t,x) = \alpha u(t,x) )</td>
<td>( W(k,h) = \alpha U(k,h) )</td>
</tr>
<tr>
<td>( w(t,x) = t^m x^n )</td>
<td>( W(k,h) = \delta(k-m)\delta(h-n) )</td>
</tr>
<tr>
<td>( w(t,x) = \frac{\partial u(t,x)}{\partial t} \frac{\partial v(t,x)}{\partial x} )</td>
<td>( W(k,h) = \sum_{i=0}^{k} \sum_{j=0}^{h} (i+1)(k-i+1)U(i+1,s-j)V(k-i+1,j) )</td>
</tr>
<tr>
<td>( w(t,x) = x^m e^{(at)} )</td>
<td>( W(k,h) = \frac{\alpha^h}{h!} \delta(k-m) )</td>
</tr>
</tbody>
</table>
\[ w(t, x) = x^m \sin(at + b) \]
\[ W(k, h) = \frac{a^h}{h!} \delta(k - m) \sin\left(\frac{h\pi}{2} + b\right) \]
\[ w(t, x) = x^m \cos(at + b) \]
\[ W(k, h) = \frac{a^h}{h!} \delta(k - m) \cos\left(\frac{h\pi}{2} + b\right) \]

**Examples and method’s applications:**

**Example (4.1):** Consider two-dimensional nonlinear wave equation with the initial condition are given by [26]

\[ \frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 u}{\partial t^2} = 1 - \frac{x^2 + t^2}{2}, \quad 0 \leq x, t \leq 1, \quad (4.21) \]

\[ u(t, 0) = \frac{t^2}{2}, \quad \frac{\partial}{\partial x} u(t, 0) = 0 \quad (4.22) \]

Taking two-dimensional differential transform of equation (4.21), we have

\[ (h + 1)(h + 2). U(k, h + 2) \]
\[ = \sum_{s=0}^{h} \left( \sum_{r=0}^{k} U(r, h - s). \sum_{r=0}^{k} U(r, h - s). (k - r + 1)(k - r + 2)U(k - r + 2, s) \right) \]
\[ + \delta(h - 0). \delta(k - 0) - \frac{1}{2} \delta(k - 2). \delta(h - 0), \quad (4.23) \]

where \( U(k, h) \) is the differential transform of \( u(t, x) \).

With transformed initial conditions equation (4.22), would be

\[ u(i, 1) = 0, \quad i = 0, 1, 2, 3, ..., n \quad (4.24) \]

Substituting equations (4.24), (4.25) into equation (4.19), and by recursive method, we obtain the following

\[ u(0, 2) = \frac{1}{2} \]

And the other of \( U(k, h) \) are zero.

Substituting Eq. (4.20), we obtain the series solution as follows

\[ u(t, x) = \frac{x^2}{2} + \frac{t^2}{2} \quad (4.25) \]
**Example (4.2):** Consider two-dimensional nonlinear wave equation with the initial condition are given by [27]

\[
\frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 u}{\partial t^2} = -t^2 \sin(t) - 2t^2 \sin^2(x), 0 \leq x, t \leq 1,
\]

(4.26)

\[u(t, 0) = \frac{t^2}{2}, \quad \frac{\partial}{\partial x} u(t, 0) = 0\]

(4.27)

Taking the differential transform of equation (4.27), we have

\[(h + 1)(h + 2)U(k, h + 2) = \sum_{s=0}^{h} \left( \sum_{r=0}^{k} U(r, h - s) \cdot (k - r + 1)(k - r + 2)U(k - r + 2, s) \right) - \]

\[\sum_{s=0}^{h} \left( \sum_{r=0}^{k} \frac{\delta(s - 0)\delta(k - r)\delta(r - 2)\sin\left(\frac{(h - s)\pi}{2}\right)}{(h - s)!} \right) \]

\[\sum_{s=0}^{h} \left( \sum_{r=0}^{k} \frac{2^{(h-s)}\delta(r - 2)\delta k - r)\delta(s - 0)\cos\left(\frac{(h - s)\pi}{2}\right)}{(h - s)!} \right) - \]

\[\delta(k - 2)\delta(h - 0), \quad k, h = 0, 1, 2, ..., N \quad (4.28)\]

Where \(U(k, h)\)is the differential transform of \(u(t, x)\).

Taking the differential transform from initial condition Eq. (4.28), and by recursive method, would be

\[U(i, 0) = 0, \quad i = 0, 1, 2, 3, 4, ..., m \quad (4.29)\]

\[U(2,1) = 1, \quad U(i, 1) = 0, i = 0, 1, 3, ..., n \quad (4.30)\]

Substituting equations (4.29), (4.30) into equation (4.28) the results are obtained as follow

\[U(2,3) = -\frac{1}{6}\]

\[U(2,5) = \frac{1}{120}\]

\[U(2,7) = -\frac{1}{5040}\]
Substituting equation (4.20), we obtain the series solution as follows

\[ u(x, t) = t^2 x - \frac{x^3 t^2}{3!} + \frac{x^5 t^2}{5!} - \frac{x^7 t^2}{7!} + \cdots \]
Chapter Five

CONCLUSIONS

The Differential Transformation Method (DTM) has been successfully applied to find exact and approximate solution of the first, second and third linear differential equations. The method was used in a direct way without using linearization, perturbation or restrictive assumptions. Therefore, it is not affected by computation round off errors. This method unlike most numerical techniques provides a closed-form solution. This technique is useful to solve linear and nonlinear differential Equation and the results obtained by it are in the Taylor’s series form. These numerical examples have proved good results.

it is implemented to the Lane-Emden differential equations as singular initial value problems. Three equations are solved and exact solutions are obtained. It is shown that differential transformation method is a very fast convergent, precise and cost efficient tool for solving the Lane-Emden equations.

We extended and proved the theorem for n-th order boundary value problems of m-th order linear differential equation.

We presented a reliable treatment based on the DTM to solve the well-known Michaelis–Menten nonlinear reaction system.

We successfully applied two- dimensional transformation method (2D DTM) for obtaining approximate solution of nonlinear differential wave equation. And the results achieved by these methods are excellent agreement with the exact solution of each example. We expect that these methods to two- dimensional nonlinear wave equations will be useful in solving other two- dimensional nonlinear equations. Also by DTM we can solve second order complex differential equation system by simple way. We can solve the system of differential equation by using DTM without taking many time.
References


