Using Infinite Fourier Sine and Cosine Transforms for Solving Linear Partial Differential Equations

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A dissertation
Submitted to the University of Gezira in Partial Fulfillment of the Requirements for the Award of the Degree of Master of Science in Mathematics

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April 2018
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DEDICATION

This work is dedicated with great love to

My Family,

Mother, Father

and

Brothers, Sister,

All Teachers and Friends
ACKNOWLEDGEMENTS

First of all my thanks to Allah who gives me all this successes and ,I truly thanks my main supervisor Dr. Abedelrahim Bashir Hamid for his continuous advice and support during writing this dissertation. Great thanks to all my. Also I am grateful thank to Dr. Altayeb Abd Elgadir to be my second supervisor for more support and encouragements. friends for their great help and support

Date: April 2018
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Salwa Altay Aahmad Mohamma

Abstract

Fourier transforms are integral ones, We get from Fourier integral treatments. Integral transforms are very important applications. The greater significance of Fourier transforms in general is in their great ability to get solutions of the partial differential equations and the ordinary ones; they also facilitate getting solutions of many integral equations. Fourier transforms depend on Fourier series which have a great importance in approximating functions in form of infinite series from sine and cosine functions at the same time. Or representing each one individually. Fourier infinite transforms in both types: there is what is called Fourier sine and cosine transforms and their reverse transforms. Moreover, that helped to create a great scientific revolution in solving differential equations, particularly, the partial differential equations, besides the questions with initial and marginal conditions. Partial differential equations have many applications in the different sciences, particularly in physics where physical questions appear when they are formulated mathematically in form of partial differential equations accompanied with initial and marginal conditions (physical conditions). Its importance is in giving solutions to these questions that are characterized with clarity and accuracy to fulfill those physical conditions.

The study aims to get accurate, easy and quick solutions for the partial differential equations by applying Fourier infinite sine and cosine transforms where the partial differential equation is transformed into ordinary differential equation to be solved as an ordinary differential equation according to its type. The study discussed how to solve Laplace equations and other as an application of the partial differential equations by using Fourier sine and cosine transforms, this gives an easy and accurate solution of physics equations. The descriptive inductive method was adopted. Among the most important results revealed by the study are: how to approximate functions in form of infinite series from the sine and cosine functions. As shown in this study that Fourier infinite sine and cosine transforms are capable of providing distinct solutions for the differential equations in an easy simple way. Thus, the study recommends the necessity of expansion in using Fourier infinite sine and cosine transforms to get accurate solutions particularly in the questions of physical nature.
استخدام تحويلات الجيب وجيب التمام غير المنتهية لفورير لحل المعادلات التفاضلية الجزئية الخطية

ملخص الدراسة

التحويلات فورير هي تحويلات تكاملية نحصل عليها من معاملات متسلسلات فورير إذ تعتبر التحويلات التكاملية من التطبيقات المهمة جداً، والأهمية العظمى لتحويلات فورير بصفة عامة هي قدرتها الهائلة على الحصول على حلول المعادلات التفاضلية الجزئية والعادية، وكذلك تسهيل الحصول على حلول الكثير من المعادلات التكاملية. وتحويلات فورير تعتمد على متسلسلات فورير وهذه المتسلسلات لها أهمية كبيرة في ترتيب الدوال على شكل متسلسلات لا نهائية من دوال الجيب وجيب التمام في أن واحد وهي ما يسمى بالتوافقية أو تمثل أيها منها على أنفده. تحويلات فورير غير المحدودة في كل التواريخ هناك ما يسمى بتحويلات الجيب، وجيب التمام لفورير وتحويلاتهم العكسية، أيضاً ساعد ذلك في إحداث ثورة علمية هائلة في حل المعادلات التفاضلية على الأخص المعادلات التفاضلية الجزئية والمسائل ذات الشروط الابتدائية والمسائل ذات الشروط الحدية. للمعادلات التفاضلية الجزئية تطبيقات عدة في مختلف العلوم وخاصة في علم الفيزياء حيث تظهر المسائل الفيزيائية عندما تصاغ صياغة رياضية على شكل معادلات تفاضلية جزئية مصحوبة بشروط ابتدائية وحدية (شروط فيزيائية) تكم أهميتها في أنها تعطي حلول لهذه المسائل تحقق تلك الشروط الفيزيائية في كلها واضح ولدقة، هدفت الدراسة إلى الحصول على حلول دقيقة وسهلة وسريعة للمعادلات التفاضلية الجزئية في تطبيق تحويلات الجيب وجيب التمام غير المنتهية لفورير حيث تم تحويل المعادلة التفاضلية الجزئية إلى معادلة تفاضلية عادية لحل بعد ذلك كمعادلة تفاضلية عادية حسب نوعها، تناولت الدراسة كيفية حل معادلات لابلاس وغيرها كتطبيق للمعادلات التفاضلية الجزئية باستخدام تحويلات الجيب، وجيب التمام لفورير يعطي حلًا سهلاً ودقيقًا لمسائل الفيزياء التي تظهر كمعادلات تفاضلية جزئية. اتبعت في هذه الدراسة المنهج الوصفي والاستقرائي، ومن أهم النتائج التي توصلت إليها الدراسة كيفية تقريب الدوال على شكل متسلسلات لا نهائية من دوال الجيب وجيب التمام كما تبين خلال هذه الدراسة أن تحويلات الجيب، وجيب التمام غير المنتهية لفورير قادرة على تقديم حل مميز للمعادلات التفاضلية بطريقة سهلة وبسيطة لذلك نوصي بالتوسع في استخدام تحويلات الجيب، وجيب التمام غير المنتهية لفورير من أجل الحصول على حلول دقيقة خاصة في المسائل ذات الطبيعة الفيزيائية التي تحتاج إلى حلول دقيقة وسهلة وسريعة.
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CHAPTER One

Introduction

1.1 General Introduction

The concept of an infinite series dates back as far as the ancient Greeks such as Archimedes (287-212 BC), who summed a geometric series in order to compute the area under a parabolic arc. The eighteenth century, power series expansions for functions like $e^x$, $\sin x$, and $\arctan x$ were first published by the Scottish mathematician Maclaurin (1698-1746), and British mathematician B. Taylor (1685-1731) generalized this work by providing power series expansions about some point other than $x = 0$. By the middle of the eighteenth century it became important to study the possibility of representing a given function by infinite series other than power series. D. Bernoulli (1700-1783) showed that the mathematical conditions imposed by physical considerations in solving the vibrating-string problem were formally satisfied by functions represented as infinite series involving sinusoidal functions. In the early 1800s, the French physicist J. Fourier came across similar representations and announced in his work on heat conduction that an "arbitrary function" could be expanded in a series of sinusoidal functions. Some of Fourier's work lacked rigor, but nevertheless provided the first real impetus to the subject now bearing his name. The Fourier integral was also first introduced by Fourier as an attempt to generalize his results from finite intervals to infinite intervals. The Fourier transform, while appearing in some early writings of A. L. Cauchy (1789-1857) and P. S. de Laplace (1749-1827), also appears in the work of Fourier. In this chapter we will discuss Fourier integral representations and Fourier transforms, [3]. In his work on heat conduction that an "arbitrary function" could be expanded in a series of sinusoidal functions. Some of Fourier's work lacked rigor, but never less be provided the first real impetus to the subject now bearing his name. The Fourier integral was also first introduced by Fourier as an attempt to generalize his results from finite intervals to infinite intervals. The Fourier transform, while appearing in some early writings of A. L. Cauchy (1789-1857) and P. S. de Laplace (1749-1827), also appears in the work of Fourier. In this chapter we will discuss Fourier integral representations and Fourier transforms, So far in this text we have studied and used only one integral transform: the Laplace transform. We saw that the Fourier integral had three alternative forms: the cosine integral, the sine integral, and the complex or exponential form. In the present section we shall take these three forms
of the Fourier integral and develop them into three new integral transforms, not surprisingly called Fourier transforms. In addition, we shall expand on the concept of a transform pair, that is, an integral transform and its inverse. We shall also see that the inverse of an integral transform is itself another integral transform [9].

Many linear boundary value and initial value problems in applied mathematics, mathematical transform, the Fourier cosine transform, or the Fourier sine transform. These transforms are very useful for the following reasons. First, these equations are replaced by simple algebraic equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green’s functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems. We begin this chapter with a formal derivation of the Fourier integral formulas. These results are then used to define the Fourier, Fourier cosine, and Fourier sine transforms. This is followed by a detailed discussion of the basic operational properties of these transforms with examples. Special attention is given to convolution and its main properties. Deal with applications of the Fourier transform to the solution of ordinary solved by the use of the Fourier transform method. The technique that is developed in this and other sections can be applied with little or no modification to different kinds of initial and boundary value problems that are encountered in applications. The Fourier cosine and sine transforms are introduced. The properties and applications of these transforms are discussed in. This is followed by evaluation of definite integrals with the aid of Fourier transforms is devoted to applications of Fourier transforms in mathematical statistics [13]. The linear superposition methods to solutions of in terms of eigenfunctions or Green’s functions. More precisely, the eigenfunction expansion method expresses the solution as an infinite series, whereas the integral solution can be obtained by integral superposition or by using Green’s functions with initial and boundary conditions. over eigenfunction expansion. First, an integral representation provides a direct way of describing the general analytical structure of a solution that may be obscured by an infinite series representation. Second, from a practical point of view, the evaluation of a solution from an integral representation may prove simpler than finding the sum of an infinite series, particularly near rapidly-varying
features of a function, where the convergence of an eigenfunction expansion is expected to be slow. Third, in view of the Gibbs phenomenon discussed in Chapter 6, the integral representation seems to be less stringent requirements on the functions that describe the initial conditions or the values of a solution are required to assume on a given boundary than expansions based on eigenfunctions. Integral transform methods are found to be very useful for finding solutions of initial and/or boundary-value problems governed by partial differential equations for the following reason. The different equation can readily be replaced by algebraic equations that are inverted by the inverse transform so that the solution of the different equations can then be obtained in terms of the original variables. The aim of this chapter is to provide an introduction to the use of integral transform methods for students of applied mathematics, physics, and engineering. Since our major interest is the application of integral transforms, no attempt will be made to discuss the basic results and theorems relating to transforms in their general forms. The present treatment is restricted to classes of functions which usually occur in physical and engineering applications [12].

1.2 Problem Inedification and Just Fication:
Methods Another for solving linear partial differential equations Method difficult.

1.3 Objectives the Research
The study aims to get accurate, easy and quick solutions for the partial differential by applying Fourier unfinished sine and cosine transformation.
Chapter Two

Literature Review

(2.1) Concept of Fourier Series in Details:

2.1.1 Linear Algebra:

A key idea in linear Algebra is that a vector can be any abstract object. Take two arbitrary vectors $V$ and $W$:

$$ V = (V_1, V_2, ..., V_n) \quad W = (W_1, W_2, ..., W_n) $$

The dot product of two vectors is then the sum of the product of the individual elements making up the two vectors.

$$ V \cdot W = (V_1 W_1 + V_2 W_2 + \ldots + V_n W_n) $$

Central to the Fourier series (and consequently the discrete Fourier transform) is the fact that continuous functions can be thought of as vectors. Take two sinusoidal functions $f$ and $g$:

$$ f(x) = \sin(x) \quad g(x) = \cos(x) $$

When the dot product of these two continuous function vectors is taken it becomes an inner product with infinitely many terms, thus an integral. [10]

(2.2) Fourier Integral Representations:

An important problem in mathematical analysis is the determination of various presentations of a given function $f$, for example a particular representation may reveal information about the function that is not so obvious by another representation. In the calculus we are taught that certain functions have power series representation of the form

$$ f(x) = \sum_{n=0}^{\infty} c_n x^n $$

Where

$$ c_n = \int_{a}^{b} f^{(n)}(x) x^{n-1} dx $$

Power series such as this are useful for numerical calculations in addition to various other uses. If the function $f$ is periodic with period $2\pi$, it may have a Fourier series representation.
Where:

\[ f(x) = \frac{1}{2a_0} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right) \]  \hspace{1cm} (2.2)

\[ a_n = \frac{1}{p} \int_{-p}^{p} f(t) \cos nt \ dt \] \hspace{1cm} (2.3)

and

\[ b_n = \frac{1}{p} \int_{-p}^{p} f(t) \sin nt \ dt \] \hspace{1cm} (n=1,2,3) \hspace{1cm} (2.4)

The theory of Fourier series shows that a periodic function satisfying certain minimal requirements can be represented by the infinite sum of sinusoidal functions given in (2.2) The formal limit of this representation as the period tends to infinity can be used to introduce the notion of a Fourier integral representation. In other words, while periodic functions defined on the entire real axis have Fourier series representations, aperiodic functions similarly defined have Fourier integral representations. If \( f \) and \( f' \) are piecewise continuous functions on some interval \([-p, p]\), we say that \( f \) is piecewise smooth. If \( f \) has this property and is periodic with period \( 2p \), it has the Fourier series representation (2.2)-(2.4). To formally obtain the Fourier integral representation of \( f \) from this series as \( p \rightarrow 0 \), we begin by substituting the integral formulas for \( a \) , \( a_n \) , and \( b_n \) given by (2.3) and (2.4) into the Fourier series (2.2). This action leads to:

\[ f(x) = \int_{-\frac{p}{2}}^{\frac{p}{2}} f(t) \cos nt \ dt + \sum_{n=1}^{\infty} \frac{1}{p} \int_{-p}^{p} f(t) \sin nt \ dt \]  \hspace{1cm} (2.5)

OR

\[ f(x) = \frac{1}{2p} \int_{-p}^{p} f(t) \ dt + \frac{1}{2p} \int_{-p}^{p} f(t) \sum_{n=1}^{\infty} \cos (n-1)x \ /p \sin nx/p \ dt \]  \hspace{1cm} (2.5)

Where we have interchanged the order of summation and integration and used the trigonometric identity [3].

\[ \cos A \cos B + \sin A \sin B = \cos (A-B) \]

We now wish to examine what happens as we let \( p \) tend to infinity. First, we must make the additional requirement that \( f \) is absolutely integrable, i.e,

\[ \int_{-\infty}^{\infty} f(t) \ dt \]  \hspace{1cm} (2.6)

\[ \lim_{p \rightarrow \infty} \frac{1}{p} \int_{-p}^{p} f(t) \ dt = \lim_{p \rightarrow \infty} \frac{1}{p} \int_{-p}^{p} f(t) \ dt \]  \hspace{1cm} (2.7)
For the remaining infinite sum in (2.5), it is convenient to let $\Delta s = \Delta x$ and then consider the equivalent limit.

$$F(x) = \lim_{\Delta x \to 0} \int_{-\infty}^{\infty} f(t) \sum_{n=1}^{\infty} \cos \left[ n(t-x) \Delta s \right] dt$$

(2.8)

(Observe that as $\Delta s \to 0$ as $p \to o$. When $s$ is a small positive number, the points are equally spaced along the axis. In such a case we may expect the series in (2.8) to approximate the integral $L''$. \[ \int_{0}^{\infty} f(t) \cos [s(t-x)] \, ds \]

In the limit as $\Delta s \to 0$. While this does not mean that the limit of the series in (2.8) is defined to be the above, we may take, under appropriate conditions on $f$ that (2.8) tends to the integral form

$$F(x) = \frac{1}{4} \int_{-\infty}^{\infty} f(t) \int_{0}^{\infty} \cos [s(t-x)] \, ds \, dt$$

(2.9)

Upon switching the order of integration, we get the equivalent form

$$F(x) = \frac{1}{4} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(t) \cos [(t-x)] \, dt \, ds$$

(2.10)

The purely formal procedure we just went through (since the passage to the limit cannot be rigorously justified) has led us to an important result known as Fourier's integral theorem [3].

(2.3) Fourier Sine and Cosine Integral Representation:

If the function $F$ is an even function, i.e., if $f(-x) = f(x)$, it follows from properties of integrals that [1].

$$A(s) = \frac{1}{4} \int_{-\infty}^{\infty} f(x) \cos (sx) \, dx = \frac{1}{2} \int_{0}^{\infty} f(x).(sx)dx$$

(2.11)

And

$$B(s) = \frac{1}{4} \int_{-\infty}^{\infty} f(x) \sin (sx) \, dx = 0$$

(2.12)

From which we deduce

$$\tilde{f}(x) = \int_{0}^{\infty} A(S) \cos (sx) \, ds$$

(2.13)

We refer to as a Fourier cosine integral representation. In a similar manner, if $f$ is an odd function, i.e., $f(-x) = -f(x)$, we obtain the Fourier sine integral representation
\[ f(x) = \int B(s) \sin(sx) \, ds \]  \hspace{1cm} (2.14)

Where \( A(s) = 0 \) and

\[ B(s) = \frac{2}{\pi} \int_{0}^{\infty} f(x) \sin(sx) \, dx \]  \hspace{1cm} (2.15)

Finally, if \( f \) should be a function defined only on the interval \( 0 < x < \infty \), we can represent it over his interval by either a Fourier cosine integral or a Fourier sine integral.

**2.4 Fourier Integral Representations:**

Used Fourier integral formula

\[ F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\varepsilon) e^{i\varepsilon x} \, d\varepsilon \right] e^{ikx} \, dk \]

To give a formal definition of the Fourier transform. [3]

**Definition (2.1):**

The Fourier transform of \( f(x) \) is denoted by \( F = \{ f(x) \} F(K) \in R \) and defined by the integral.

\[ F \{ f(x) \} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\varepsilon x} \, dx \]  \hspace{1cm} (2.16)

the factor \( \frac{1}{\sqrt{2\pi}} \) involved in where \( F \) is called the Fourier transform operator or the Fourier transformation and the factor \( \frac{1}{\sqrt{2\pi}} \) is obtained by splitting

\[ f(x) = F(k) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\varepsilon) e^{i\varepsilon x} \, d\varepsilon \right] e^{ikx} \, dk \]  \hspace{1cm} (2.17)

is often called the complex Fourier transform. A sufficient condition for \( f(x) \) to have a Fourier transform is that \( f(x) \) is absolutely integrable on \( (-\infty, \infty) \). The convergence of the integral (1.16) follows at once from the fact that \( f(x) \) is absolutely integrable. In fact, the integral converges uniformly with respect to \( k \). Physically, the Fourier transform \( F(k) \) can be interpreted as an integral superposition of an infinite number of sinusoidal oscillations with different wave numbers \( k \) or different wavelengths \( \lambda = 2\pi k \). Thus, the definition of the Fourier transform is restricted to absolutely integrable functions. This restriction is too strong for many physical applications. Many simple and common
functions, such as constant function, trigonometric functions $\sin(ax)$, $\cos(ax)$, exponential functions, and $x_n H(x)$ do not have Fourier transforms, even though they occur frequently in applications. The integral in (1.16) fails to converge when $f(x)$ is one of the above elementary functions. This is a very unsatisfactory feature of the theory of Fourier transforms. However, this unsatisfactory feature can be resolved by means of a natural extension of the definition of the Fourier transform of a generalized function, $f(x)$ in (1.16). We follow Lighthill (1958) and Jones (1982) to discuss briefly the theory of the Fourier transforms of good functions. The inverse Fourier transform, denoted by $F^{-1}\{F(k)\} = f(x)$, is defined by $F^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} F(k) dk$, (1.17) where $F^{-1}$ is called the inverse Fourier transform operator. Clearly, both $F$ and $F^{-1}$ are linear integral operators. In applied mathematics, $x$ usually represents a space variable and $(k = 2\pi\lambda)$ is a wave number variable where $\lambda$ is the wavelength. However, in electrical engineering, $x$ is replaced by the time variable $t$ and $k$ is replaced by the frequency variable ($\omega = 2\pi\nu$) where $\nu$ is the frequency in cycles per second. The function $F(\omega) = F\{f(t)\}$ is called the spectrum of the time signal function $f(t)$. In electrical engineering literature, the Fourier transform pairs are defined slightly differently by [12].

$$F\{f(t)\} = F(V) = \int_{-\infty}^{\infty} f(t) e^{-2\pi it}$$ \hspace{1cm} (2.18)

And

$$F^{-1}\{f(v)\} = f(t) = \int_{-\infty}^{\infty} f(v) e^{2\pi ivt} dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t}$$ \hspace{1cm} (2.19)

Where ($\omega = 2\pi\nu$) is called the angular frequency. The Fourier integral formula implies that any function of time $f(t)$ that has a Fourier transform can be equally specified by its spectrum. Physically, the signal $f(t)$ is represented as an integral superposition of an infinite number of sinusoidal oscillations with different frequencies $\omega$ and complex amplitudes $\frac{1}{2\pi} F(\omega)$. Equation (2.18) is called the spectral resolution of the signal $f(t)$, and $F(\omega) = 2\pi$ is called the spectral density. In summary, the Fourier transform maps a function (or signal) of time $t$ to a function of frequency $\omega$. In the same way as the Fourier series expansion of a periodic function decomposes the function into harmonic
components, the Fourier transform generates a function (or signal) of a continuous variable whose value represents the frequency content of the original signal. This led to the successful use of the Fourier transform to analyze the form of time-varying signals in electrical engineering and seismology (Or [12]).

(2.5) Fourier Transform:

We first give a formal definition of the Fourier transform by using the complex Fourier integral formula [12].

Definition (2.2)

If \( u(x,t) \) is a continuous, piecewise smooth, and absolutely integrable function, then the Fourier transform of \( u(x,t) \) with respect to \( x \in \mathbb{R} \) is denoted by \( u(k,t) \) and is defined by:

\[
F \{ u(x,t) \} = u(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} \, dx \tag{2.20}
\]

where \( k \) is called the Fourier transform variable and \( e^{-ikx} \) is called the kernel of the transform. Then, for all \( x \in \mathbb{R} \), the inverse Fourier transform of \( u(k,t) \) is defined by

\[
F^{-1} u(k,t) = u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} u(k,t) \, dk \tag{2.21}
\]

We may note that the factor \( \frac{1}{2\pi} \) in the Fourier integral formula has been split and placed in front of the integrals (2.20) and (2.21). Often the factor \( \frac{1}{2\pi} \) can be placed in only one of the relations (2.20) and (2.21). It is not uncommon to adopt the kernel \( e^{ikx} \) in (1) instead of \( e^{ikp} \), and as a consequence, \( e^{ikx} \) would be replaced by \( e^{ikp} \) in (2) [12].

Definition (2.3): (Fourier Transform Pairs)

\[
F \{ f(x) \} = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} \, dx = F(\alpha) \tag{2.22}
\]

Inverse Fourier transform:

\[
F^{-1} \{ F(\alpha) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, d\alpha = f(x) \tag{2.23}
\]

(ii) Fourier sine transform:

\[
\frac{1}{2\pi} \int_{0}^{\infty} f(x) \sin \alpha x \, d\alpha = f(x)
\]
\[ F_s \{ f(x) \} = \int f(x) e^{iux} dx = f(a) \]  
(2.24)

Inverse Fourier sine transform
\[ F_s^{-1} \{ F(a) \} = \int_{0}^{\infty} f(x) \sin(ax) dx = F(a) \]  
(2.25)

(iii) Fourier cosine transform:
\[ F_c \{ f(x) \} = \int_{0}^{\infty} f(x) \cos(ax) dx = F(a) \]  
(2.26)

Inverse Fourier cosine transform:
\[ F_c^{-1} \{ F(a) \} = \int_{0}^{\infty} f(a) \cos(ax) da = f(x) \]  
(2.27)

Definition (2.4):
The Inversion Formula.

The Fourier transform is defined as follows
\[ f(u) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) e^{iu \epsilon} d\epsilon \]  
(1)

For brevity, we rewrite formula (1) as follows:
\[ f(u) = F\{ f(x) \} \text{ or } f(u) = F\{ f(x), u \} \]

Given \( f(u) \) the function \( f(x) \) can be found by means of the inverse Fourier transform.
\[ F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(U) e^{iu \epsilon} du \]  
(2)

Formula (2) hold for continuous functions. If \( f(x) \) has (a finite) Jump discontinuity at a point \( x = x_0 \), then the left-hand side of (2) is equal to \( \frac{1}{2}[f(x_0^-) + f(x_0^+)] \) at this point.

For brevity, we rewrite formula (2) as follows:
\[ F(u) = F\{ f(x) \} \text{ or } F(u) = F\{ f(x), u \} \]

Definition (2.5):

The convolution of two integrable functions \( f(x) \) and \( g(x) \), denoted by \( (f \ast g)(x) \), is defined by.
\[ (f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \epsilon) g(\epsilon) d\epsilon \]

Provided the integral in
\[ (f \ast g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \epsilon) g(\epsilon) d\epsilon \]
Exists, where the factor is a matter of choice. In the study of convolution, this factor is often omitted as this factor does not affect the properties of convolution. We will include or exclude the factor freely.

**Definition (2.6)**:

Suppose a real or complex valued function $g(x)$ is defined for all $x \in \mathbb{R}$ and is infinitely differentiable everywhere, and suppose that each derivative tends to zero as $|x| \to \infty$ faster than any positive power of $(x-1)$ or in other words, suppose that for each positive integer $N$ and $n$,

$$\lim_{X \to \infty} X^N g^{(n)}(x) = 0,$$

then $g(x)$ is called a good function [13].

**Definition (2.7)**:

Under the assumptions on $f(x)$ similar to those made for the one-dimensional case, the multiple Fourier transform of $f(x)$ where $x=(x_1, x_2, \ldots, x_n)$ is the n-dimensional vector, is defined by [3].

$$F\{f(x)\} = \mathcal{F}(K) = \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{-i(k \cdot x)\} f(x) \, dx.$$  

Where $k = (k_1, k_2, \ldots, k_n)$ is the n-dimensional transform vector and $k \cdot x = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n$. The inverse Fourier transform is similarly defined by

$$F^{-1}\{F(k)\} = f(x) = \left( \frac{1}{(2\pi)^{\frac{n}{2}}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{-i(k \cdot x)\} F(k) \, dk.$$  

In particular, the double Fourier transform is defined by

$$F\{f(x,y)\} = f(x,y) = \frac{1}{2^l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{-i(k \cdot r)\} F(k,l) \, dk \, dl.$$  

Where $r = (x,y)$ and $k=(k,l)$. The inverse Fourier transform is given by

$$F^{-1}\{F(k,l)\} = f(x,y) = \frac{1}{2^l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{i(k \cdot r)\} F(k,l) \, dk \, dl.$$  

Similarly, the three-dimensional Fourier transform and its inverse are defined by the integrals

$$F\{f(x,y,z)\} = f(k,m) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{-i(k \cdot r)\} f(x,y,z) \, dx \, dy \, dz$$  

$$F^{-1}\{F(k,m)\} = f(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{i(k \cdot r)\} f(k,m) \, dk \, dl \, dm.$$  

The operational properties of these multiple Fourier transforms are similar to those of the one-dimensional case. In particular.

$$
\begin{array}{c|c|c|c}
\infty & \infty & 0 \\
-\infty & 0 & -\infty
\end{array}
\begin{cases}
-1 & x < 0 \\
1 & x > 0
\end{cases}$$
\[
\int s \ g n(x)dx = \int g(x) \cdot g(x) \ dx
\]

Relating the Fourier transforms of derivatives to the Fourier transforms of given functions are valid for the higher-dimensional case as well. In higher dimensions, they are applied to the transforms of partial derivatives of \( f(x) \) under the assumptions that \( f \) and its partial derivatives vanish at infinity. We illustrate the multiple Fourier transform method by the following examples of applications [13].

**2.6 Categories of Fourier Transform Formations:**

The general term: Fourier transform, can be broken into four categories, resulting form the four basic types of signals that can be encountered. A signal can be either continuous or discrete and it can be either periodic or aperiodic.

The family of Fourier transform Fourier analysis is named after Jean Baptiste Joseph Fourier (1768-1830), a French mathematician and physicist (Fourier is pronounced : and is always capitalized).

While many contributed to the field, Fourier is honored for his mathematical discoveries and in sight in to the practical use fullness of the techniques. Fourier was interested in heat propagation, presented a paper in 1807 to the Institut d'France on the use of sinusoids to represent temperature distributions. The paper contained the controversial claim that any continuous periodic signal could be represented as the sum of famous mathematicians Joseph Louis Lagrange (173-1813), and Pierre Simon Laplace (1749-1827).

While Laplace and the other reviewers voted to publish the paper Lagrange adamantly protested. For nearly 5 years, Lagrange had insisted that such an approach could not be used to represent signals with corners, i.e., discontinuous slopes, such as in square waves. The Institute d'France bowed to the prestige of Lagrange, and rejected Fourier's work. It was only after Lagrange's death, when the paper was finally published, more than 15 years later, that Fourier had other things to keep him busy. Political activities (expedition to Egypt with Napoleon and trying to avoid the guillotine after the French Revolution (literally)). Who was right? It was correct in his assertion that a summation of sinusoids can not form a signal with a corner how ever; you can get very close that the difference between the two has zero energy. In this sense, Fourier was...
right, although 18th century science knew little about the concept of energy. This phenomenon now goes by the name: Gibbs Effect, and will be discussed [10].

(2.7) Summary Of Fourier Transforms:
In these notes, we have studied four different kinds of Fourier transforms:
1. The continuous-time Fourier transform (C T F T)
2. The discrete-time Fourier transform (D T F T)
3. The continuous time Fourier series (C T F S)
4. The discrete Fourier transform (D F T) OR discrete Fourier series (D F S)
The last three transforms can be considered as special cases of the (CTFT) obtained by sampling (multiplying by a periodic impulse train) in either the time domain or the frequency domain, or both. Sampling in one domain corresponds to an aliasing operation (convolution by a periodic impulse train). To each of these transforms correspond both a convolution theorem and a product theorem. Convolutions can be either discrete or continuous, and either cyclic or (linear). Which of these four types of convolutions should be used follows from the properties of the signals to be convoluted: discrete convolution for discrete signals, cyclic convolution for periodic signals, and convolution modulo N for signals that are both discrete and periodic [10].

(2.8) Fourier Transform:
Suppose that \( f \) is continuous and absolutely integrable on the interval \((\infty, -\infty)\) and \( f^{-1} \)
is piecewise continuous on every finite interval. If \( f(x) \rightarrow 0 \) as \( x \rightarrow \pm \infty \), then integration by parts gives:

\[
F \{ f^{-1}(x) \} = \int_{\infty}^{-\infty} f^{-1}(x) e^{-iax} dx = f(x) e^{iax} - ia \int_{\infty}^{-\infty} f(x) e^{iax} dx = -ia \int_{\infty}^{\infty} f(x) e^{-iax} dx,
\]

\[
F \{ f^{-1}(x) \} = -ia F(a)
\]

That is. Similarly, under the added assumptions that \( f^{-1} \) is continuous one very finite interval and \( f^{-1}(x) \rightarrow 0 \) as \( x \rightarrow \pm \infty \), we have

\[
F \{ f^{-1}(x) \} = ( -ia )^2. F \{ f^{-1}(x) \} = -a^2 F(a)
\]

It is important to be aware that the sine and cosine transforms are not suitable for transforming the fiderivative (or, for that matter, any derivative of odd order). It is readily shown that

\[
F_s \{ f^{-1}(x) \} = - a F_c \{ f(x) \} \text{ and } F_s \{ f(x) \} f(0)
\]
The difficulty is apparent; the transform of \( f^- (x) \) is not expressed in terms of the original integral transform [12].

(2.9) **Fourier Transform Pairs:**

Fourier' integral theorem. If \( f \) and \( f^- \) are piece wise continuous function on every finite interval and if

\[
\int_{-\infty}^{\infty} |f(x)| dx < \infty
\]

States that

\[
\int_{0}^{\infty} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos[s(t-x)] dt \, ds \quad (2.1)
\]

Through the use of Euler's formula, \( \cos x = \left( e^{ix} + e^{-ix} \right) \), we can express (1.) in terms of complex exponential functions. That is,

\[
f(x) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \left( e^{-istx} + e^{-istx} \right) dt \, ds = \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-istx} \, dt \, ds
\]

Or

\[
f(x) = \frac{1}{2\pi} \int_{0}^{\infty} e^{-ix} \int_{0}^{\infty} e^{ix} f(t) \, dt \, ds \quad (2.2)
\]

Which is the exponential form of Fourier's integral theorem. What we have established by the integral formula (2) is the pair of transform formulas [3].

\[
F(s) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{isx} f(t) \, dt \quad (2.3)
\]

And

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-isx} F(s) \, ds \quad (2.4)
\]

We define \( F(s) \) as the Fourier transform of \( f(t) \) also written as

\[
F(s) = \{ f(t) ; s \} \quad (2.5)
\]

And \( f(t) \) as the inverse Fourier transform of \( F(s) \), which may be written as

\[
F(t) = F^{-1} \{ (F(s) ; t) \} \quad (2.6)
\]

The location of the constant \( \sqrt{2\pi} \) in the definition of the transform pairs is arbitrarily selected as long as (2) is satisfied. For reasons of symmetry we have split the constant
between the transform pairs, but in the literature no universal agreement exists on the location of these constants. In some texts, the constant \( \frac{1}{\sqrt{2\pi}} \) is positioned in front of one of the transform pairs with no constant in front of the other. There is also some variation as to which integral represents the transform and which one represents the inverse transform. In practice, of course, these differences are of little consequence but the user should be aware of them when consulting different reference sources. As an immediate consequence of (3), we observe that [10].

\[
|F(s)| \leq \frac{1}{\sqrt{2\pi}} \int |f(t)| \, dt
\]

if \( f \) is absolutely integrable it follows that its transform function \( F(s) \) is bounded. A similar argument applied to (4) shows that \( f(t) \) is also bounded when \( F(s) \) is absolutely integrable. Furthermore, the Riemann-Lebesgue Lemma shows that

\[
\lim_{|s| \to \infty} F(s) = 0
\]

Since the transform function \( F(s) \) associated with absolutely integrable functions that are also piecewise smooth must satisfy this last relation, it immediately rules out certain functions as possible transform functions. For example, sine, cosines, and polynomials do not satisfy this relation. Finally, it is a curious property that although the function \( f(t) \) may have certain finite discontinuities, its transform \( F(s) \) can be shown to be a continuous function. Because of this, the Fourier transform is sometimes called a "smoothing process" [13].

**2.10 Fourier Sine Transform:**

Suppose that \( f \) and \( f^- \) are continuous, \( f \) is absolutely integrable on the interval \([0, \infty)\)

And \( f^- \) is piecewise continuous on every finite interval. If \( f \to 0 \) and \( f^- \to 0 \) as \( x \to 0 \), then

\[
F_s\{ f^- \} = \int_0^\infty f(x) \sin ax \, dx - a \int_0^\infty f(x) \cos ax \, dx = -a [ \int_0^\infty f(x) \cos ax \, dx] + a[f(x) \sin ax \, dx] = a f(0) - a^2 f_0[f(x)]
\]

That is,

\[
F_s\{ f^- \} = a^2 F(a) + a F(a)(0)
\]
(2.11) Fourier Cosine Transform:

Under the same assumptions that lead we find the Fourier cosine transform of \( f(x) \)
To be

\[
F_c \{ f'(x) \} = a^2 F(a) + a f(0)
\]

A natural question is “How do we know which transform to use on a given boundary-
value problem?” Clearly, to use a Fourier transform, the domain of the variable to be
eliminated must be \((-\infty, \infty)\)

. To utilize a sine or cosine transform, the domain of at least one of the variables in the
problem must be \((0, \infty)\).

But the determining factor in choosing between the sine transform and the cosine
transform is the type of boundary condition specified at zero. In the examples that
follow, we shall assume without further mention that both \( u \) and. \[12\]

(2.12) Fourier Cosine and Sine Transform:

The Fourier cosine integral formula

\[
f(x) = f(-x) = \frac{2}{\pi} \int_0^\infty f(\varepsilon) \cos k \varepsilon d\varepsilon
\]

This is called Fourier cosine integral formula

\[
f(x) = f(-x) = \frac{2}{\pi} \int_0^\infty \sin k x \int_0^\infty f(\varepsilon) \sin k \varepsilon d\varepsilon
\]

leads to the Fourier cosine trans- form and its inverse defined by

\[
\{ F_c f(x) \} = F_c(k) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \cos k x f(x) dx,
\]

\[
\{ F_c^{-1} f(k) \} = F(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \cos k sF(k) dk,
\]

Where \( F_c \) is where \( F_c \) is the Fourier cosine transform operator and \( F_c^{-1} \) is its inverse operator.

Similarly, the Fourier sine integral formula leads to the Fourier sine transform and its
inverse defined by

\[
F_s \{ f(x) \} = F_s(k) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \sin k f(x) dx,
\]

\[
F_s^{-1} \{ F_s(k) \} = f(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \sin k sF_s(k) dk,
\]

Where \( F_s \) is the Fourier sine transform operator and \( F_s^{-1} \) is its inverse [12].
(2.13) **Properties OF Fourier Transform:**

**Theorem (2.1):**

(linearity) The Fourier transformation F is linear.

Proof: we have

\[
F_c |f(x)| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx,
\]

Then, for any constants a and b,

\[
F | af(x + bg(x)) | = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |af(x) + bg(x)| e \, da \, dx,
\]

**Theorem (2.2) (Shifting)**

Let \( F | f(x) | \) be a Fourier transform of \( f(x) \).

\[
F | f(x-c) | = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F | f(c-t) | e^{ikx} \, dx
\]

\[
F_c^{-1} F_s |f(x)| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\varepsilon) \ e^{-ikx} \, d\varepsilon \text{ where } \varepsilon = x - c = e^{-ikx} F|f(x)|,
\]

Where \( c \) is a real constant.

Proof: Form the definition, we have for \( c > 0 \),

**Example (2.1)**

Find the solution of the Dirichlet problem in the half-plane \( y > 0 \)

\[
u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0
\]

\[
U(x,0) = f(x), \quad -\infty < x < \infty
\]

\( U \) and \( u_x \) vanish as \( |x| \rightarrow \infty \), and \( u \) is bounded as \( y \rightarrow \infty \).

Let \( u(k,y) \) be the Fourier transform of \( u(x,y) \) with respect to \( x \).

Then

\[
u(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-ikx} \, dx
\]

Application of the Fourier transform with respect to \( x \) gives

\[
u_{yy} - k^2 = 0
\]

\( u(k,0) = F(k) \) and \( u(k,y) \quad 0 \rightarrow \text{asy} \rightarrow \infty
\]

The solution of this transformed system is

\[
u(k,y) = F(k) e^{-|k|y}
\]

The inverse Fourier transform of \( u(k,y) \) gives the solution in the form

\[
u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\varepsilon) e^{-i|k|\varepsilon} \, d\varepsilon \right] e^{-ikx} \, dk
\]
It follows from proof of example
\[ \int_{-\infty}^{\infty} e^{-ik(t-x)} |k|y \, dk = \frac{2y}{(x-y)^2 + y^2} \]
Hence the solution of the Dirichlet problem in the half-plane \( y > 0 \)
Is:
\[ u(x,y) = \frac{y}{2\pi} \frac{f(\epsilon)}{(x-\epsilon)^2 + y^2} \, d\epsilon \]
From this solution, we can readily deduce a solution of the Neumann problem in the half-plane \( y > 0 \) [3].

**Theorem (2.3) : (Scaling).**
If \( F \) is the Fourier transform of \( f \), then
\[ F[f(cx)| = \frac{1}{|c|} F[f(k/c) \]
Where \( c \) is a real non-zero constant
Proof : For \( c \neq 0 \)
\[ F[f(cx)| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(cx) \, e^{-ikx} \, dx \]
If we let \( \epsilon = cx \), then
\[ F[f(cx)| = \frac{1}{|c|} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\epsilon) \, e^{-i(k/c)\epsilon} \, d\epsilon = \left( \frac{1}{|c|} \right) F\left( \frac{k}{c} \right) \] [3].

**Theorem (2.4) : (Differentiation).**
Let \( f \) be continuous and piecewise smooth in \((-\infty, \infty)\). Let \( f(x) \) approach zero as \( |x| \to \infty \).
If \( f \) and \( f' \) are absolutely integrable, then
\[ F[f''(x)] = ikF[f(x)] \, ikF(k) \]
Proof:
\[ F[f''(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f''(x) \, e^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}} \left[ f(x) \, e^{-ikx} \right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) \, e^{-ikx} \, dx \]
\[ = F[ikF[f(x)] \, ikF(k)] \]
This result can be easily extended. If \( f \) and its first \((n-1)\) derivatives are continuous, and if its \( n \)th derivative is piecewise continuous, then 0
\[ = F[f^n(x)] = (ik)^n F[f(x)] = (ik)^n F(k), n= 0, 1, 2, \ldots \]
provided and its derivatives are absolutely integrable. In addition, we assume that \( f \) and its first \((n-1)\) derivatives tend to zero as \( |x| \) tends to infinity. If \( u(x,t) \to 0 \) as \( x \to 0 \), then
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \, dx \]
\[ F\left( \frac{\partial u}{\partial x} \right) = \int e^{ix} u(x,t) \, dx, \]

Which is obtained by parts

\[ = \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} e^{-ikx} u(x,t) \, dx + \int_{-\infty}^{\infty} e^{-ikx} u(x,t) \, dx \right] \]

\[ ikF\{ u(x,t) \} = iku(k,t) \]

Similarly, if \( u(x,t) \) is continuously \( n \) times differentiable and \( \frac{\partial^m u}{\partial x^m} \to 0 \) as \( |x| \to 0 \) for \( m = 1, 2, 3, \ldots, (n-1) \) then

\[ = F\{ \frac{\partial^n u}{\partial x^n} \} = (ik)^n F\{ u(x,t) \} = (ik)^n u(k,t). \]

It also follows from the definition that

\[ F\left( \frac{\partial^n u}{\partial x^n} \right) = \frac{du}{dt}, F\left( \frac{\partial^n u}{\partial t^n} \right) = \frac{du}{dt}, \ldots, F\left( \frac{\partial^n u}{\partial t^n} \right) = \frac{du}{dt^n}. \]

The definition of the Fourier transform shows that a sufficient condition for \( u(x,t) \) to have a Fourier transform is that \( u(x,t) \) is absolutely integrable in \( -\infty < x < \infty \). This existence condition is too strong for many practical applications. Many simple functions, such as a constant function, \( \sin(\omega x) \), and \( x^m H(x) \), do not have Fourier transforms even though they occur frequently in applications. The above definition of the Fourier transform has been extended for a more general class of functions to include the above and other functions. We simply state the fact that there is a sense, useful in practical applications, in which the above stated functions and many others do have Fourier transforms [13].

**Theorem (2.5)** : (Convolution Theorem):

If \( F(k) \) and \( G(k) \) are the Fourier transform of \( f(x) \) and \( g(x) \) respectively, then the Fourier transform of the convolution \( (f * g) \) is the product \( F(k) G(k) \). That is,

\[ F(f(x), g(x)) = F(k)G(k) \]

Or, equivalently

\[ F^{-1}\{ F(k)G(k) \} f(x), g(x) \]

More explicitly

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)G(k) e^{-ikx} \, dk = (f * g)(x) \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\epsilon) \, g(\epsilon) \, d\epsilon \]

**Proof** : By definition, we have

\[ = F(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} f(x-\epsilon) \, g(\epsilon) \, d\epsilon \, d\epsilon \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} f(x-\epsilon) \, g(\epsilon) \, d\epsilon \, d\epsilon \]
\[
= \frac{1}{2\sqrt{\pi}} \int g(\xi) \, d\xi \int f(x - \xi) e^{-i k(x - \xi)} \, dx
\]

With the change of variable \( \eta = x - \xi \), we have

\[
= \mathcal{F} \{ f \ast g \} = \frac{1}{\sqrt{2\pi}} \int f(\eta) e^{ik\eta} \, d\eta = \frac{1}{\sqrt{2\pi}} \int f(\eta) \, e^{ik\eta} \, d\eta = F(k)F(k),
\]

The convolution satisfies the following properties

1. \( F \ast g = g \ast f \) (commutative).
2. \( F \ast (g \ast h) = (F \ast g) \ast h \) (associative)
3. \( F \ast (a g \ast bh) = a(f \ast g) + b (f \ast g) \) (distributive) [3]

**Theorem (2.6)**

The Fourier transform of a good function is a good function.

**Proof:**

The Fourier transform of a good function \( f(x) \) exists and is given by

\[
F \{ f(x) \} = F(k) = \frac{1}{\sqrt{2\pi}} \int f(x) \, e^{ikx} \, dx
\]

Differentiating \( F(k) \) \( n \) times and integrating \( N \) times by parts we get

\[
|F^n| \leq \left( \frac{1}{\sqrt{2\pi}} \right)^n \int e^{ikx} \left| \frac{d^N}{dx^N} (\xi^N f(x)) \right| dx \leq \frac{1}{|k|^N} \int \left| \frac{d^N}{dx^N} (x^N f(x)) \right| dx
\]

Evidently, all derivatives tend to zero as fast as \( |k|^N \) as \( |k| \to \infty \) for any \( N > 0 \) and hence, \( F(k) \) is a good function [3].

**Theorem (2.7):**

If \( f(x) \) is a good function with the Fourier transform

\[
F \{ f(x) \} = F(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) \, dx
\]

then the inverse Fourier transform is given by

\[
f(x) = \int_{-\infty}^{\infty} e^{-ikx} F(k) \, dk
\]

**Proof:**

For any \( \epsilon > 0 \) we have [3].

\[
F \{ e^{i \epsilon x} F(-x) \} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{ikx} \left\{ \int_{-\infty}^{\infty} e^{-ikt} f(t) \, dt \right\} dx
\]

**Theorem (2.8):**

If \( f(x) \) is piecewise continuously differentiable and absolutely integrable, then

1. \( F(k) \) is bounded for \(-\infty < k < \infty \)
(ii) $F(k)$ is continuous for $-\infty < k < \infty$

Proof:

It follows for the definition that

$$|F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \, dx = \frac{c}{\sqrt{2\pi}}$$

Where $c = \int_{-\infty}^{\infty} |f(x)| \, dx = \text{constant}$. This proves result (i).

To prove (ii), we have

$$|F(k + h) - F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} |f(x)| \, dx \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \, dx$$

Since $\lim_{h \to 0} e^{-ikx} = 0$ for all $x \in \mathbb{R}$, we obtain

$$\lim_{h \to 0} |F(k + h) - F(k)| = 0$$

This shows that $F(k)$ is continuous.

Theorem (2.9): (Riemann-Lebesgue lemma)

If $F(k) = Ff(x)$, then $\lim_{|k| \to \infty} |F(k)| = 0$

Proof: since $e^{-ikx} = e^{-ikx}$, we have

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x - \frac{1}{k}) \, dx$$

Hence,

$$F(k) = \frac{1}{2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx \cdot \int_{-\infty}^{\infty} e^{-ikx} f(x - \frac{1}{k}) \, dx \right\} = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \cdot f(x - \frac{1}{k}) \, dx$$

There for,

$$|F(k)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x) f(x - \frac{1}{k})| \, dx$$

Thus, we obtain

$$\lim_{|k| \to \infty} |F(k)| \leq \frac{1}{\sqrt{2\pi}} \lim_{|k| \to \infty} \int_{-\infty}^{\infty} |f(x) f(x - \frac{1}{k})| \, dx = 0$$
Chapter three

Material and Method

(3.1) Study Site

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(3.2) Material and Method

The basic data for this research are collected from books, journals, reports and official documents.

The research methodology is the scientific inductive style.

(3.3) Fourier Cosine & Sine Transforms

We found that when the function $f$ is even, the Fourier integral representation of $f(x)$ reduces to.

$$f(x) = \int_0^{\infty} A(s) \cos sx \, ds = \frac{2}{\pi} \int_0^{\infty} f(t) \cos st \, dt \, ds$$

Based on this relation we introduce the Fourier cosine transform

$$F_c \{ f(t); s \} = \frac{\sqrt{\pi}}{\sqrt{2}} \int_0^{\infty} \cos st \, dt \, ds = f_c(s), s > 0$$

And inverse cosine transform

$$F_c^{-1} \{ f(t); t \} = \frac{\sqrt{\pi}}{\sqrt{2}} \int_0^{\infty} F_c^{-1}(s) \cos( s) \, ds = f(t), t > 0$$

These results are interesting in that they imply the equivalence of the operators $F_c$ and $F_c^{-1}$. In other words, the cosine transform and its inverse are exactly the same in functional form. Similarly, when $f$ is an odd function its Fourier integral representation

Which leads to the Fourier sine transform

$$F_c^{-1} \{ f(t); s \} = \frac{\sqrt{\pi}}{\sqrt{2}} \int_0^{\infty} f(t) \sin st \, dt \, ds = F(s), s > 0$$

And inverse sine transform

$$F_c^{-1} \{ F(s); t \} = \frac{\sqrt{\pi}}{\sqrt{2}} \int_0^{\infty} F_c(s) \sin s \, ds = f(t), t > 0$$

Hence, we see that the Fourier sine transform and its inverse are also exactly the same in functional form. If the function $f$ is neither even nor odd, but defined only for $t > 0$, 


then it may have both a cosine transform and a sine transform. Moreover, the even and odd extensions of \( f \) will then have exponential Fourier transforms. To see the relations between these various transforms, let us construct the even extension of \( f \) by setting

\[
f_c(t) = f(|t|) - \infty < t < \infty
\]

The Fourier transform from of \( f_c(t) \) leads to

\[
F\{f_c(t)\; t\} = \left[ \frac{\pi}{i} \right] \int_{-\infty}^{\infty} f_c(t) e^{-ist} dt
\]

\[
= \left[ \frac{\pi}{i} \right] \int_{0}^{\infty} f_c(t) \cos st dt + \left[ \frac{\pi}{i} \right] \int_{0}^{\infty} f_c(t) \sin st dt
\]

Form which we deduce

\[
F\{f(t); s\} = F\{f_c(t); s\} - \infty < s < \infty (1)
\]

Based on (1), it is clear that the Fourier transform and cosine transform of an even function give identical results. In particular, their transforms are even functions of \( s \).

The odd extension of \( f \) is constructed by setting

\[
f_0(t) = f(|t|) \text{sgn(t)}, -\infty < t < \infty
\]

Where the signum function is defined by

\[
\text{sgn}(t) = \begin{cases} 
1, & t > 0 \\
-1, & t < 0 \\
0, & t = 0 
\end{cases}
\]

in this case, we find

\[
F_0\{f_0(t); s\} = \left[ \frac{\pi}{i} \right] \int_{0}^{\infty} f_0(t) e^{-ist} dt
\]

\[
= \left[ \frac{\pi}{i} \right] \int_{0}^{\infty} f_0(t) \cos st dt + \left[ \frac{\pi}{i} \right] \int_{0}^{\infty} f_0(t) \sin st dt
\]

Because the Fourier transform of an odd function is also an odd function

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin \frac{2\pi n}{p} dt , n=1,2,3,...
\]

We make the conclusion that the Fourier transform and sine transform are related by

\[
F\{f_c(t); s\} = F_s\{f(t) ; (|s|)\} s g n(s), -\infty < s < \infty (2)
\]

The practical use of is that if we want to evaluate the Fourier transform of an even function, we can do so by simply calculating its cosine transform. If the function we wish to transform is odd, we first can find its sine transform and then use (2). These observations are very useful when using tables to find transforms, since most of the known transforms are either cosine or sine transforms. A short table of transforms is presented [3].
(3.4) Fourier Sine Transform:
Suppose that $f$ and $f'$ are continuous, $f$ is absolutely integrable on the interval $[0, \infty)$, and $f$ is piecewise continuous on every finite interval. If $f \to 0$ and $f' \to 0$ as $x \to \infty$, then,

$$F( f'(x)) = \int_{0}^{\infty} f'(x) \sin ax \, dx = f(x) \sin ax \left[ \int_{0}^{\infty} f'(x) \cos ax \, dx \right]$$

$$= a \left[ f(x) \sin ax + \int_{0}^{\infty} f(x) \sin ax \, dx \right]$$

$$= a f(0) - a^2 F_s( f(x))$$

That is $F_s( f'(x)) = -a^2 F(a) + a f(0)$

(3.5) Fourier Cosine Transform:
Under the same assumptions that lead to we find the Fourier cosine transform of $f(x)$ to be

$$F_c( a f'(0)) = -a^2 F(a) + a f'(0)$$

A natural question is "How do we know which transform to use on a given boundary-value problem?" Clearly, to use a Fourier transform, the domain of the variable to be eliminated must be $(-\infty, \infty)$.

To utilize a sine or cosine transform, the domain of at least one of the variables in the problem must be $[0, \infty)$. But the determining factor in choosing between the sine transform and the cosine transform is the type of boundary condition specified at zero [3].

The Fourier Sine Transform:
let a function $f(x)$ be enterable on the semiaxis $0 \leq x < \infty$ the Fourier sine transform defined by

$$f_s'(u) = F_s( f(x))$$

It follows from formula let a function $f(x)$ be integrable on the semiaxis $0 \leq x < \infty$ the Fourier sine transform is defined by

$$f_s'(u) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \sin(ux) \, dx \quad 0 < u < \infty$$

(3.4)

For given $f_s'(u)$ The function $f(x)$ can be found by means of the inverse Fourier sin transform.
\[ f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f_s(u) \sin(xu) \, du \quad 0 < x < \infty \] (3.5)

The Fourier sine transform (4) is briefly) that the Fourier sine transform has the property equation.

1. Three tables of the Fourier sine transform. Which are sue solving special equations

Sometimes it is more convenient to apply the asymmetric form of the Fourier sine transform defined by the following two formulas

\[ f_s(u) = \int_{0}^{\infty} f(x) \sin(xu) \, dx, \quad \hat{f}_s(x) = \int_{0}^{\infty} f_s(u) \sin(xu) \, du \] (3.6)

The direct and inverse Fourier sine transform forms (6) are denoted by

\[ f_s(u)(u) = f_s f(x) and f(x) = F_s^{-1} \{ f_s^{-1}(u) \}, \text{ respectively.} \]

(3.6) Examples

Example (3.1) Find the Fourier sine and cosine transform \( s \) of \( te, \ a > 0 \)

Solution:

Formal differentiation of both sides of

\[ I = \int_{0}^{\infty} e^{-at} \, dt = \frac{3}{S^2 + a^2} \quad a > 0 \]

First with respect to \( a \) and then with respect to \( s \), gives us respectively,

\[ - \int_{0}^{\infty} te^{-at} \, dt = \frac{-2as}{(S^2 + a^2)^2} \]

And

\[ - \int_{0}^{\infty} te^{cost} \, dt = \frac{(a^2 - s^2)}{(S^2 + a^2)^2} \]

Thus, we deduce that

\[ F_s \{ te^{-at} \} = \frac{2as}{(S^2 + a^2)^2} \quad a > 0 \]

And

\[ F_s \{ te^{cost} \} = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{(a^2 - s^2)}{(S^2 + a^2)^4} \quad a > 0 \]

Example (3.2)

Find the Fourier sine transform of \( \frac{1}{2} e^{-at}, \ a > 0 \).

Solution:

We begin by integrating both sides of

\[ \int_{0}^{\infty} e^{-at} \, dt = \frac{1}{s} \quad 0 < s < \infty \]
\[ I = \int e \sin st \, dt = a > 0 \]

With respect to parameter \( a \) from \( \frac{1}{a} \) to \( \infty \), which leads to
\[ I = \int_{0}^{\infty} e^{-at} \, \sin st \, dt = \int_{0}^{\infty} \frac{3}{S^2 + a^2} \, da = \frac{1}{2} \tan^{-1} \frac{a}{S} \]

Thus, it follows that
\[ F_s \left\{ \frac{1}{i} \, e^{-at}; s \right\} = \frac{1}{\sqrt{\pi}} \tan^{-1} \frac{s}{a} \quad a > 0 \]

If we allow \( a \to 0^+ \) in the result of example (2), we find
\[ F_s \left\{ \frac{1}{i} \, e^{-at}; s \right\} = \frac{1}{\sqrt{\pi}} \quad (1) \]

This result is only a formal result since neither \( \frac{1}{i} \) nor \( \frac{1}{\sqrt{\pi}} \) satisfy the conditions of the Fourier integral theorem. Nonetheless, it can be useful to treat (1) as a limiting case of the transform relation given in Example (2) using
\[ F_s \left\{ f(t); s \right\} = iF_s \left\{ f(t); |s|s \right\} s \quad g(n), -\infty < s < \infty \]

We obtain the similar relation
\[ F \left\{ \frac{1}{i} \, e^{st}; s \right\} = i \frac{1}{\sqrt{\pi}} s \quad g(n), \quad (2) \]

Finally, as a bonus, we see that (2) provides a generalization
\[ \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \sin \frac{t}{s} \, ds = \]

Which is
\[ \int_{0}^{\infty} \sin^{-1} \, dt = \frac{1}{2} s \quad g(n), \]

**Example (3.3):**

show that
\[ (a) F_c \left\{ \frac{a}{x} \, e^{x}; a > 0 \right\} = \frac{1}{\sqrt{\pi}} \frac{s}{(a^2 + k^2)} \]
\[ (a) F_c \left\{ \frac{a}{x} \, e^{x}; a > 0 \right\} = \frac{1}{\sqrt{\pi}} \frac{k}{(a^2 + k^2)} \]

We have
\[ F_c \left\{ e^{\frac{ak}{x}} \right\} = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-akx} \, dx = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-ikx} + e^{ikx} \, dx \]

The proof of the other result is similar and hence left to the reader.

Using the above results with the Fourier cosine and sine inverse transformations and an antitraduction of variables, we find that
\[ \frac{1}{(x^2 + a^2)^{\frac{1}{2}}} \frac{e^{ax}}{a} \]
According to the Fourier cosine and sine inverse transformations, we write

\[ e^{-ak} = \frac{2a}{\pi} \int_{0}^{\infty} \cos \frac{kx}{a^2 + x^2} \, dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{x \sin \frac{kx}{a^2 + x^2}}{a^2 + x^2} \, dx \]

Interchanging x and k these results become

\[ e^{-ak} = \frac{2a}{\pi} \int_{0}^{\infty} \cos \frac{xk}{a^2 + x^2} \, dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{x \sin \frac{xk}{a^2 + x^2}}{a^2 + x^2} \, dx \]

Thus, it follows that

\[ F_c \left\{ \frac{1}{(x^2 + a^2)} \right\} = \frac{\pi}{\sqrt{a}} \int_{0}^{\infty} \cos \frac{xk}{a^2 + x^2} \, dx = \frac{\pi}{\sqrt{a}} e^{-ak} \frac{x}{a^2 + x^2} \]

\[ F_s \left\{ \frac{x}{(x^2 + a^2)} \right\} = \frac{\pi}{\sqrt{a}} \int_{0}^{\infty} \frac{x \sin \frac{xk}{a^2 + x^2}}{a^2 + x^2} \, dx = \frac{\pi}{\sqrt{a}} e^{-ak} \frac{x}{a^2 + x^2} \]

**Example (3.4):**

Show that

\[ F_s^{-1} \left\{ \frac{1}{x} \exp (-sk) \right\} = \frac{\pi}{\sqrt{a}} \tan^{-1} \left( \frac{x}{s} \right) \]

We have the stand and definite integral

\[ \frac{\pi}{\sqrt{a}} F_s^{-1} \left\{ \frac{1}{x} \exp (-sk) \right\} = \int_{0}^{\infty} \exp (-sk) \sin x \, dk = \frac{x}{(x^2 + a^2)} \]

Integrating both sides with respect to s from s to \( \infty \) gives

\[ \int_{s}^{\infty} \frac{\pi}{\sqrt{a}} \sin x \, dk = \int_{s}^{\infty} \frac{xds}{(x^2 + a^2)^{3/2}} = \left[ \tan^{-1} \left( \frac{x}{s} \right) \right] = \frac{\pi}{2} - \tan^{-1} \left( \frac{x}{s} \right) \]

Thus

\[ F_s^{-1} \left\{ \frac{1}{x} \exp (-sk) \right\} = \frac{\pi}{\sqrt{a}} \int_{0}^{\infty} k \exp (-sk) \sin x \, dk = \frac{\pi}{\sqrt{a}} \tan^{-1} \left( \frac{x}{s} \right) \]

(3.7) Properties of Fourier Cosine and Sine Transforms:

**Theorem (3.1):**

If \( F_c \{ f(x) \} = F_c \{ k \} \) and \( F_s \{ f(x) \} = F_s \{ k \} \) then

\[ F_c \{ f(x) \} = \frac{1}{a} F_c \{ \frac{x}{a} \}, a > 0 \]

\[ F_s \{ f(x) \} = \frac{1}{a} F_s \{ \frac{x}{a} \}, a > 0 \]

Under appropriate conditions, the following properties also hold

\[ F_c \{ f'(x) \} = F_c \{ k \} = -\frac{\pi}{\sqrt{a}} f(0) \]

\[ F_c \{ f''(x) \} = -k^2 F_c \{ k \} = -\frac{\pi^2}{a} f'(0) \]
\[ F_c \{ f' (x) \} = -k F_c (k) \]
\[ F_s \{ f'' (x) \} = -k^2 F_s (k) + \frac{\pi}{i} k f (0) \]

These results can be generalized for the cosine and sine transforms of higher order derivatives of function. They are left as exercises [3].

**Theorem (3.2):**

(Convolution Theorem for the Fourier cosine transform).

If \( F_c^{-1} \{ F_c (k) \} = F_c (k) \) and \( F_c \{ g (x) \} = G_c (k) \) then

\[ F_s^{-1} \{ F_c (k) G_c (k) \} = \frac{\pi}{i} \int_0^\infty f (\epsilon) [ g(x + \epsilon) + g (|x - \epsilon|) ] d \epsilon \]

Or equivalently,

\[ \int_0^\infty F_c (k) G_c (k) \cos kx dk = \frac{1}{2} \int_0^\infty f (\epsilon) [ g(x + \epsilon) + g (|x - \epsilon|) ] d \epsilon \]

**Proof:**

Using the definition of the inverse Fourier cosine transform we have

\[ F_s^{-1} \{ F_c (k) G_c (k) \} = \frac{\pi}{i} \int_0^\infty F_c G_c (k) \cos kx dk \]

\[ = \int_0^\infty G_c (k) \cos kx dk \int_0^\infty f (\epsilon) \cos \epsilon d \epsilon \]

Hence

\[ F^{-1} \{ F_c (k) G_c (k) \} = \left( \frac{\pi}{i} \right) \int_0^\infty f (\epsilon) d \epsilon \int_0^\infty \cos kx \epsilon dk G_c dk \]

\[ = \frac{1}{2} \frac{\pi}{i} \int_0^\infty f (\epsilon) d \epsilon \int_0^\infty \left[ \cos (x + \epsilon) + \cos (|x - \epsilon|) \right] G_c dk \]

\[ = \frac{1}{2} \frac{\pi}{i} \int_0^\infty f (\epsilon) \left[ g(x + \epsilon) + g (|x - \epsilon|) \right] d \epsilon \]

In which the definition of the inverse Fourier cosine transform is used. This proves [3]

\[ F^{-1} \{ F_c (k) G_c (k) \} = \frac{1}{\sqrt{2}} \int_0^\infty f (\epsilon) \left[ g(x + \epsilon) + g (|x - \epsilon|) \right] d \epsilon \]

It also follows from the proof of theorem

\[ \int_0^\infty F_c^{-1} (k) G_c (k) \cos kx dk = \frac{1}{\sqrt{2}} \int_0^\infty f (\epsilon) \left[ g(x + \epsilon) + g (|x - \epsilon|) \right] d \epsilon \]

This proves result. Putting \( x = 0 \) in

\[ \int_0^\infty F_c (k) G_c (k) \cos kx dk = \frac{1}{2} \int_0^\infty f (\epsilon) \left[ g(x + \epsilon) + g (|x - \epsilon|) \right] d \epsilon \]
We obtain
\[ \int_0^\infty F_c(k) G_c(k) \, dk = \frac{1}{2} \int f(\epsilon) g(\epsilon) \, d\epsilon = \int f(x) g(x) \, dx \]

Substituting \( g(x) = f(x) \) gives, since \( G_c(k) = F_c(k) \)
\[ \int_0^\infty |F_c(k)|^2 \, dk = |f(x)|^2 \, dx \]

This is the Parseval relation for the Fourier cosine transform. Similarly, we obtain
\[ \int_0^\infty F_s(k) G_s(k) \cos kx \, dk = \int G_s(k) \cos kx \, dk \]
\[ \int_0^\infty f(\epsilon) \sin k \epsilon \, d\epsilon = \frac{1}{\sqrt{\pi}} \int_0^\infty g(\epsilon) \sin k \epsilon \, d\epsilon \]

Which is by interchanging the order of integration,
\[ \int_0^\infty f(\epsilon) \, d\epsilon \int_0^\infty G_s(k) \sin k \epsilon \, x \, dk \]
\[ = \frac{1}{\sqrt{\pi}} \int_0^\infty f(\epsilon) \, d\epsilon \int_0^\infty G_s(k) [\sin(\epsilon + x) + \sin(\epsilon - x)] \, dx \]
\[ = \frac{1}{\sqrt{\pi}} \int_0^\infty f(\epsilon) \, d\epsilon [g(\epsilon + x) + g(\epsilon - x)] \, dx \]

(1) In which the inverse Fourier sine transform is used. Thus, we find
\[ \int_0^\infty F_s(k) G_s(k) \cos kx \, dk = \frac{1}{\sqrt{\pi}} \int_0^\infty f(\epsilon) \, d\epsilon \int_0^\infty [g(\epsilon + x) + g(\epsilon - x)] \, d\epsilon \]

or, equivalently,
\[ F_s \{F_s(k) G_s(k)\} = \frac{1}{\sqrt{\pi}} \int_0^\infty f(\epsilon) \, d\epsilon \int_0^\infty [g(\epsilon + x) + g(\epsilon - x)] \, d\epsilon \]

Result or (2) is also called the convolution theorem of the Fourier cosine transform.

Putting \( x = 0 \) in (1) gives
\[ F_s \{F_s(k) G_s(k)\} \int_0^\infty f(\epsilon) g(\epsilon) \, d\epsilon = f(x) g(x) \, dx \]

Replacing \( g(x) \) by \( \overline{f(x)} \) gives the Parseval relation for the Fourier sine transform
\[ \int_0^\infty |F_s(k)|^2 \, dk \int_0^\infty |f(x)|^2 \, dx \]
Chapter Four
Results and discussion

(4.1) Results:
form through these study communicate researcher to results dependableness

(1) The main finding of the study is the most important results revealed by the study are: how to approximate functions in form of infinite serials from the sine and cosine functions. As shown in this study that Fourier unfinished sine and cosine transformations are capable of providing distinct solutions for the differential equations in an easy simple way.

(2) Acting function below some conditions to obtained for Fourier series Image series infinite form functions sine and function cosine

(3) The definition on transform she series auto reintegrates Fourier and straying transform turn auto what designate by transforms Fourier

(4) Transforms sine and cosine infinite to Fourier capable adducing solutions distinguishable to equations differentiation by method easy and simple

(5) Transform Fourier capable on transform function auto rests below ether auto function another by variable independent another and by domain definition another.

(4.2) Examples:

Example (4.2.1)

(one Dimensional Diffusion equation on a Half line).
Consider initial–boundary value problem for the one dimensional diffusion equation in 0<x<∞ with nosources or sinks:

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad 0<x<\infty, \quad t>0 \]

(4.1)

Where k is a condition , with the initial condition

\[ U_u(x,0) = 0 \quad 0<x<\infty \]

(4.2)

And the boundary conditions .

(a) \( u(x,0) = f(t), t>0, u(x,t) \to 0 \quad \text{as} \quad x \to \infty \)

(4.3)

Or,
(a) \( u_s(0,x)=f(t), \; t \geq 0, \; u(x,t) \to 0 \quad \text{as} \; x \to \infty \)

This problem with the boundary condition (3) is solved by using the Fourier sine transform.

\[
\begin{align*}
u_s(k,t) &= \int_0^\infty \sin(kx)u(x,t)\,dx, \\
\frac{du_s}{dt} &= -kK^2 u_s(k,t) + \sqrt{\frac{k}{\pi}} Kf(t),
\end{align*}
\]  \hspace{1cm} (4.6)

The bounded solution of this differential system with \( u_s(k,0)=0 \) is

\[
u_s(k,t) = \sqrt{\frac{k}{\pi}} \int_0^\infty f(t)\exp \left[-k(t-T)k^2\right] \,dT
\]

\hspace{1cm} (4.7)

The inverse transform gives the solution

\[
u(x,t) = k[-f(t) F_s^{-1}\{ (k\exp[-k(t-T)k^2]) \} ]
\]

\hspace{1cm} (4.8)

In which \( F_s^{-1}\{ (k\exp[-tkK^2])\} = \frac{\sqrt{\frac{k}{\pi}}}{kT0} \) is used in particular, \( f(t)=T_k \) constant

\[u_s(k,t) = \sqrt{\frac{k}{\pi}} \left(1 - \exp(-k^2t)\right) dx
\]

\hspace{1cm} (4.9)

Inversion gives the solution

\[
u_s(x,t) = \left(-\frac{2x^2}{nk}\right) \int_0^\infty \frac{\sin(kx)}{k} \left[1 \cdot \exp(-k^2t)\right] \,dk
\]

Making use of integral

\[
u_s(x,t) = \int_0^\infty e^{-k^2a^2} \frac{\sin(kx)}{k} \,dk = \frac{x}{\sqrt{\pi}} \text{erf}\left(\frac{x}{\sqrt{2}}\right)
\]

The solution becomes

Where the error function, \( \text{erf}(x) \) is defined by

\[
\text{erf}(x) = \int_0^x e^{-u^2} \,du, \quad u(x,t) = \left[-2\text{erf}\left(\frac{x}{\sqrt{2t}}\right)\right] = T_0 \text{erf} c \left(\frac{x}{\sqrt{2mt}}\right)
\]

So that

\[
\text{erf}(0), \quad \text{erf}(\infty) \int_0^\infty e^{-u^2} \,du = 1 \quad \text{and} \quad \text{erf}(-x) = -\text{erf}(x)
\]

and the complementary error function, \( \text{erf} c(x) \) is defined by

\[
\text{erf}(x) = 1 - \text{erf}(x) = \frac{x}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \,du
\]
So that

\[-\text{erf}(x) = 1 - \text{erf}(x) , \quad \text{erf}(0) = 1 \quad \text{erf}(\infty) = 0\]

And

\[\text{erf}(-x) = 1 - (x) = 1 + \text{erf}(x) = 2 - \text{erfc}(x)\]

Equation (1) with boundary condition (2) is solved by the Fourier cosine transform

\[u(k,t) = \int_{0}^{\infty} \cos(kx) u(x,t) dx\]

Application of this transform to (1) gives

\[\frac{dk}{dt} + k^{2} u_{c} = \frac{f(t)}{4k^{2}} \quad (4.10)\]

The solution of (3) with \(u(k,0) = 0\) is

\[u_{c}(k,y) = \frac{\pi}{2} \int_{\frac{\pi}{2}}^{\infty} f(T) \exp \left[-k^{2} (t-T)\right]dT\]

Since

\[F_{c}^{-1}\{\exp(-tk^{2})\} = \frac{1}{\sqrt{2\pi t}} \exp \left(-\frac{k^{2}}{4kt}\right)\]

The inverse Fourier cosine transform gives the final form of the solution

\[u(k,y) = \int_{\frac{\pi}{2}}^{\infty} f(T) \exp \left[-\frac{k^{2}}{4k(T-t)}\right]dT\]

**Example(4.2.2):**

(The Laplace Equation in the Quarter plane) solve the Laplace equation

\[u_{xx} + u_{yy} = 0 , \quad 0 < x , y < \infty\]

With the boundary condition s

\[u(0,y) = a, \quad u(x,0) = 0 \quad \text{as} \quad r = \sqrt{x^{2} + y^{2}} \rightarrow \infty\]

Where a is constant

We apply the Fourier sine transform with respect to x to find

\[\frac{d^{2}u_{s}}{dx^{2}} - k^{2} u_{s} + \frac{a}{2} = 0\]

The solution of this in homogeneous equation

\[u_{s}(k,y) = A e^{ky} + \frac{a}{2k} \quad ,\]

Where A is a constant to be determination from \(u(k,0) = 0\) consequently,

\[u_{s}(k,y) = \frac{a}{2k} \left( 1 - e^{-ky} \right)\]

The inverse transformation gives the formal solution

\[u_{s}(x,y) = \frac{2a}{\pi} \int_{0}^{\frac{\pi}{2}} (1 - e^{-ky}) \sin kx dx\]

Or,

\[u_{s}(x,y) = \frac{2a}{\pi} \left[ \int_{0}^{\frac{\pi}{2}} \frac{\sin kx}{k} dk - e^{-ky} \sin kx dk \right] = a - \frac{2a}{\pi} \left( \frac{\pi}{2} - \tan^{-1} \right) = \frac{2a}{\pi} \tan^{-1} \left( \frac{\pi}{2} \right)\]
In which
\[ \frac{\pi}{2} \tan^{-1} \left( \frac{x}{y} \right) = \tan^{-1} \left( \frac{x}{y} \right) \]

**Example: (4.2.3)**

(The Laplace equation in a semi–infinite with the Dirichlet data) solve the Laplace equation

\[ u_{xx} + u_{yy} = 0 , \quad 0 < x < \infty \quad 0 < y < b \quad (1) \]

With the boundary conditions

\[ u(0,y) = 0 , \quad u(x,y) \to 0 \text{ as } x \to \infty \text{ for } 0 < y < b \quad (2) \]

\[ u(x,b) = 0 , \quad u(x,0) = f(x) \text{ for } 0 < x < \infty \quad (3) \]

In view of the Dirichlet data , the Fourier sine transform with respect to x can be used to solve this problem. Applying the Fourier sine transform to (1) - (3) gives

\[ \frac{du_s}{dy^2} - k^2 u_s = 0 \quad (4) \]

\[ u_s(k,b) = 0 , \quad u_s(k,0) = F_s(k) \quad (5) \]

The solution of (4) with (5) is

\[ u_s(x,y) = F_s(k) \frac{\sinh[k(b-y)]}{\sinh[kb]} \]

The inverse Fourier sine transform gives the formal solution

\[ u_s(k,b) = \frac{\pi}{2} \int_0^\infty \int_0^\infty \left( f(l) \sin kldl \right) \frac{\sinh[k(b-y)]}{\sinh[kb]} \]

In the limit as \( kb \to \infty \),

\[ \approx \exp(-ky) \text{hence the above problem reduces to the corresponding problem in the quarter plane} 0 < x < \infty , 0 < y < \infty \quad \text{thus} \quad \text{solution(1) becomes} \]

\[ u(x,y) = \frac{\pi}{2} \int_0^\infty \left( f(l) \sin x \sinh y \right) \exp(-ky)dk \]

\[ = \frac{1}{\pi} \left( f(l) \sin x \sinh y \right) \int_0^\infty \left( \cos(x-1) - \cos(x+1) \right) \exp(-ky)dk \]

\[ = \frac{1}{\pi} \left( f(l) \right) \left[ \frac{y}{(x-1)^2 + y^2} - \frac{y}{(x+1)^2 + y^2} \right] dl \]

This is the exact integral solution of the problem if \( f(x) \) is an odd function of \( x \), then solution (1) reduces to solution

Of the same problem in the half plane
Example: (4.2.4)

Using the cosine transform The steady state temperature in a semi-infinite plate is determined from

\[
\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \quad 0 < x < \pi
\]

\[
u(x,y) = 0, \quad u(\pi, y) = e^y, y > 0
\]

Solve for \(u(x,y)\)

Solution: The domain of the variable \(y\) and the prescribed condition at \(y = 0\) indicate that the Fourier cosine transform is suitable for the problem. We define

\[
F_c\{u(x,y)\} = \int_0^\infty u(x,y)\cos(ny)\,dy = \hat{u}(x,a)
\]

In view of \(F_c\{f'(x)\} = a^2 F(a) - f'(0)\)

\[
F_c\{\frac{d^2u}{dx^2}\} + \{\frac{d^2u}{dy^2}\} = F_c(0)
\]

Becomes \(\frac{d^2u}{dx^2} + a^2 u(x,a) - u_y(x,0) = 0\), or \(\frac{d^2u}{dx^2} a^2 = 0\)

Since the domain of \(x\) is a finite interval we choose to write the solution of ordinary differential equation as

\[
u(x,a) = c_1 \cosh ax + c_2 \sinh ax
\]

Now \(F_c\{u(0,y)\} = F_c(0)\) and \(F_c\{u(\pi,y)\}\)

\(F_c\{e^y\}\) are in turn equivalent

\[
u(\pi,a) = 0, \quad u(x,0) = \frac{1}{1+a^2}
\]

When we apply these latter the solution

\[
u(x,a) = c_1 \cosh ax + c_2 \sinh ax
\]

Gives \(c_1 = 0\) and \(\frac{C_2 - 1}{(1+a^2)\sinh \pi}\)

Therefore

\[
u(x,a) = \frac{\sinh x}{(1+a^2)\sinh \pi}
\]

So from \(F^{-1}\{F(a)\} = \frac{2}{\pi} \int_0^\infty f(a) \cos(ax)\,dx = f(x)\)

We a write at

\[
u(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\sin x}{(1+a^2)\sinh \pi}\cos ydx
\]

Had \(u(x,0)\) been given in example (2) rather than \(u(x,0)\) then the sine transform would have been a appropriate.
Example (4.2.5)

Show that
\[ \int_0^a x^p \, dx = \frac{\pi}{2} \, \text{sec} \left( \frac{\pi}{2} \right) \]

We write
\[ f(x) = e^{-ax} \rightarrow F_c(k) = \frac{\pi}{2} \frac{a}{(a^2 + x^2)} \]
\[ g(x) = x^p \rightarrow G_c(k) = \frac{\pi}{2} k^p \langle p \rangle \cos \left( \frac{\pi}{2} \right) \]

Using Parseval result for the Fourier cosine transform gives
\[ \int_0^\infty F_c(k) \, G_c(k) \, dk = \int_0^a f(x) \, g(x) \, dx \]

Or
\[ \frac{2a}{\pi} \cos \left( \frac{\pi}{2} \right) = \int_0^\infty \frac{k^p \langle p \rangle}{(k^2 + a^2)} \, dx \]
\[ = \frac{1}{ap} \int_0^a e^{-t} \, t^p \, dt = \frac{\langle p \rangle}{ap} \, (ax=t) \]

Thus,
\[ \frac{\pi}{2} \frac{k^p \langle p \rangle}{(k^2 + a^2)} = \frac{\pi}{2} \frac{e^{a+b}}{2(a+b)} \]

Example (4.2.6)

If \( a > 0 \), \( b > 0 \), show that
\[ \int_0^a \frac{x^2 \, dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2(a+b)} \]

We consider
\[ F_s\{ e^{-ax} \} = \frac{\pi}{2} \frac{k}{(k^2 + a^2)} = F_s(k) \]
\[ F_s\{ e^{-bx} \} = \frac{\pi}{2} \frac{k}{(k^2 + b^2)} = F_s(k) \]

Then the convolution theorem for the Fourier cosine transform gives
\[ \int_0^a F_s(k) G\cos kx \, dk \]
\[ = \int_0^a g(x) \, f(x + x) + f(x - x) \, dx \]

Putting \( x = 0 \) gives
\[ \int_0^a F_s(k) G_s(k) \, dk = \int_0^a g(x) \, f(x) \, dx \]

Or,
\[ \int_0^a \frac{k^2 \, dk}{(k^2 + a^2)(k^2 + b^2)} = \frac{\pi}{2} \int_0^a e^{(a+b)x} \, dx = \frac{\pi}{2(a+b)} \]

Example (4.2.7)

Show that
\[ \int_0^a \frac{x^p \, dx}{(a^2 + x^2)} = \frac{\pi}{2(2a)} \langle p \rangle \, , a > 0 \]

We write \( f(x) = \frac{1}{2(a^2 + x^2)} \) so that
\[ \int_0^a \frac{x^p \, dx}{(k^2 + a^2)^{\pi/2}} = \frac{\pi}{2(2a)} \langle p \rangle \, , a > 0 \]
\[ f'(x) = \quad \text{and} \quad F\{f(x)\} = F(k) \]
\[ = \frac{1}{2a} \left( \frac{x}{a} \right) \exp (-a|k|) \]

Making reference to the parseral relation we obtain

\[ | f'(x)|^2 dx = |F| | f'(x)|^2 dk \]
\[ | ik|^2 F(f(x))^2 dk \]
\[ = \frac{\pi}{(2a)^2} K^2 \exp (-2ak) dk = \frac{\pi}{(2a)^2} \]

This gives the desired result

**Example (4.2.8)**

Show that

\[ \int_0^x e^{-a+b|x|^2} dx = \frac{\pi}{\sqrt{a+b}} a>0, b>0 \]
We write \( f(x) = e^{ax^2} \) and \( g(x) = e^{bx^2} \)
So that \( F(k) = \frac{1}{\sqrt{2a}} e^{\frac{k^2}{2a}} \) and \( G(k) = e^{ck^2} \)and the use the formula

To obtain

\[ \int_0^x f(x) = g (-x) \ dx = \frac{1}{\sqrt{2ab}} \int_0^\infty e^{-\frac{k^2}{2ab}} \left( 1 + \frac{1}{a+b} \right) \ dk \]
\[ = \frac{\pi}{\sqrt{2ab}} \int_0^\infty e^{\frac{ck^2}{2}} \ dk = \frac{\pi}{\sqrt{a+b}} \]

**Example (4.2.9)**

Find the temperature distribution in a semi-in finite rod for the following cases with zero initial temperature distribution

(a) The heat supplied at the end \( x=0 \) at the rate \( g(t) \);
(b) The end \( x = 0 \) is kept at a constant temperature \( T_0 \).

The problem here is to solve the conduction equation

\[ u_t = ku_{xx} \quad x>0, t>0 \]
\[ u(x,0) = 0, x>0 \]

(a)\( u_x(0,t) = g(t) \) and (b)\( u(0,t) = T_0 \)

Here we assume that \( u(x, t) \)and \( u_x(0,t) \) vanish as \( x \to \infty \) for case (a) ,let \( u(k,t) \) be the Fourier cosine transform of \( u(x, t) \)

Then the transformation of the heat conduction equation yields

\[ u_t + kk^2 u = - \left( \frac{\pi}{\sqrt{\pi}} \right) g(t)k \]

The solution of this equation with \( u(k,0) = 0 \) is

\[ \int_0^\infty \left( \frac{\pi}{\sqrt{\pi}} \right) g(t)k \]
\[
\begin{align*}
  u(x, t) &= u(k, t) \cos kx \, dk \\
  &= \frac{-2k}{\pi} \int_0^\pi e^{-k^2k(t-T)} \cos kx \, dk
\end{align*}
\]

The inner integral is given by(s)

\[
\int_0^\pi e^{-k^2k(t-T)} \cos kx \, dk = \frac{1}{2} \left( \frac{\pi}{\sqrt{t(t-T)}} \right) - \exp \left( -\frac{x^2}{4k(t-T)} \right)
\]

The solution, there for is

\[
u(x, t) = \left[ \frac{k}{\sqrt{\pi}} \right] \int_0^l \frac{g(t)}{\sqrt{t(t-T)}} \cdot e^{-x^2/4k(t-T)} \, dt
\]

For case (b), we apply the Fourier sine transform of \( u(x, t) \) to obtain the transformed equation

\[
u_t + kK^2 u = \frac{1}{\pi^2} kT_0 \frac{1}{k} \cdot e^{-ak^2}
\]

The solution of this equation with zero initial conditions is

\[u(k, t) = T_0 \left( \frac{2\pi}{k} \right) \cdot 1 - e^{-ak^2}
\]

Then the inverse Fourier sine transformation gives

\[u(x, t) = \frac{2T_0}{\pi} \int_0^\pi \sin \frac{kx}{k} \left( 1 - e^{-ik^2} \right) \, dk
\]

Making use of the integral

\[e^{-a^2} \frac{\sin \frac{kx}{k}}{k} \, dk = \frac{\pi}{2} \cdot \text{erf} \left( \frac{x}{2\sqrt{t}} \right)
\]

The solution is found to be

\[u(x, t) = \frac{\pi}{2} \frac{2T_0}{\pi} \left[ \frac{\pi}{2} \cdot \text{erf} \left( \frac{x}{2\sqrt{t}} \right) \right]
\]

\[= T_0 \cdot \text{erf} \left( \frac{x}{2\sqrt{t}} \right)
\]

Where \( \text{erfc}(x) = 1 - \text{erf}(x) \) is the complementary error function defined by \( \text{erfc}(x) = \int_x^\infty e^{-t^2} \, dt \)
Chapter Five
Conclusion & Recomendation

(5.1) Conclusion

The main finding of the study is the most important results revealed by the study are: how to approximate functions in form of infinite serials from the sine and cosine functions. As shown in this study that Fourier unfinished sine and cosine transformations are capable of providing distinct solutions for the differential equations in an easy simple way.

(5.2) Recommendation:

The study recommends the necessity of expansion in using Fourier unfinished sine and cosine transformations to get accurate solutions particularly in the questions of physical nature.
References
