Technique for the Solution of Ordinary Differential equations as Integral Equations

By
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Department of Mathematics
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January, 2014
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Date: January, 2014
Technique for the Solution of Ordinary Differential equations as Integral Equations

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Date of Examination: 18, January, 2014
DEDICATION

This is work is dedicated with great love.

My Family,

To

Mother(Fatoma), Father,

Sisters, brothers,

Teachers and Friends
Acknowledgement

I would like to thank all those who had helped me in the presentation of this work.

I would like to particularly to acknowledge the guideness of our supervisor Dr. Mohamed Hassan Mohamed

Thanks to my teachers at university and staff for their help through the year of study
Technique for Solving the Differential Equations as Integral Equations

Omar Abdulazim El Mahi Ellobaid Ahmed

Abstract

This research is a simplified concept of the technique of solving ordinary differential equations as integral equations. No one can deny the effective and distinguished role performed by the integral equations in the fields of mechanics and physics where many of the threshold values in mathematical physics are returned to integral equations. The objective of this research is to device a unified method for solving ordinary linear and linear differential equations from the higher ranks. Moreover, integral equations are considered an important tool in many branches of analysis like functional analysis and random evloutionaries. This subject is considered to be a principal discipline in old and modern mathematics and also considered as an indispensible strong support for the specialists in the fields in physics, chemistry, mechanics and mathematics. In the following, we will confine all the study to the second type of the integral equations. Fredholm integral equation of the second type will have the greatest share of this study. The reason for this is that at most a person encounters this kind of integral equations when finding out solutions for threshold values problems in integral equations and mathematical physics. Then the theory of integral equations of the second type are to a larger easier than theory of integral equations of the first type. The method adopted in this research is the method of Coloumb team for solving linear and non-linear equations and some repetitive methods. Given the importance characterizing the subject of solving the integral equations, the concept of an integral equation was defined. Also some types of special equations and the methods of their solution were explored in the first and second chapters. How this equation will be solved by the core status and deteriorating core was explained. The results arrived by the study is the application of this research where in it the problems of threshold and primary values are solved for ordinary differential equations after transforming them to integral equations. Also the concept of Dirac delta and Green’s function with some examples were shown. the study recommends the device of a technique or a transform to solve the partial differential equations as integral equations.
تقنية لحل المعادلات التفاضلية العادية كمعادلات تكاملية

عمر عبد العظيم الماهي العبيد أحمد

ملخص الدراسة

هذا البحث عبارة عن مفهوم مبسط لتقنية حل المعادلات التفاضلية العادية كمعادلات تكاملية، ولايستطيع أحد ان ينكر الدور الفعال والمميز الذي تزوده المعادلات التكاملية في حلول الميكانيك والفيزياء حيث ترد كثير من مسائل القيم الحدية للفيزياء الرياضية إلى المعادلات التكاملية، والهدف من هذا البحث وضع طريقة موحدة لحل المعادلات التفاضلية العادية الخطية وغير الخطية من الرتب العليا وعلاوة على ذلك فإن المعادلات التكاملية تعد أداة هامة في كثير من فروع التحليل مثل التحليل التابعي والطوريات العشوائية، ويعتبر هذا الموضوع من المواد الرئيسية في الرياضيات، ويعتبر سندا قويا لايستغني عنه للعاملين في حقول الفيزياء والكيمياء والبيئيات والرياضيات، وسيوفر قصرا جل دراسة فيما يلي في المعادلات التكاملية من النوع الثاني، وسوف يكون لمادة فريد هولوم من النوع الثاني النصيب الأكبر من هذه الدراسة، وسبب في ذلك يعود إلى أن الأدلة يصادف على الكثير من المعادلات التكاملية عند إيجاد الحلول لمسائل القيم الحدية في المعادلات التفاضلية والفيزياء الرياضية ثم إن نظرية المعادلات التكاملية من النوع الثاني أسهل إلى حد كبير من نظرية المعادلات التكاملية من النوع الأول، والطريقة المتبعة في هذا البحث هي طريقة فريد هولوم لحل المعادلات التكاملية الخطية وغير الخطية وبعض الطرق التكرارية ونظرية للأهمية التي يتصرف بها موضوع حل المعادلات التكاملية فقد عرفت مفهوم المعادلة التكاملية وانواعها كما تعرض لدراسة بعض أنواع المعادلات الخاصة وطرق حلها فقد خصصت لها الفصل الأول والثاني وبيت كيف يتم حل هذه المعادلة بوساطة التدوين الحالة والدوبر المتردي، والنتائج التي توصلت لها هذه الدراسة هي تطبيق هذا البحث حيث يتم فيه حل لمسائل القيم الحدية والإبتدائية للمعادلات التفاضلية العادية، وذلك بعد تحويلها إلى معادلات تكاملية ثم حلها. توصي هذه الدراسة بعمل تقنية أو تحويلة لحل المعادلات التفاضلية الجزئية كمعادلات تكاملية.
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Chapter 1

INTEGRAL EQUATIONS WITH SEPARABLE KERNELS

1.1 Definition

An integral equation is an equation in which an unknown function appears under one or more integral signs. Naturally, in such an equation there can occur other terms as well. For example, for

\[ a \leq s \leq b , \quad a \leq t \leq b \]

The equations

\[ f(s) = \int_a^b k(s, t) g(t) \, dt \quad (1) \]

\[ g(s) = f(s) + \int_a^b k(s, t) g(t) \, dt \quad (2) \]

\[ g(s) = \int_a^b k(s, t) [g(t)]^2 \, dt \quad (3) \]

Where the function \( g(s) \) is the unknown function while all the other functions are known, are integral equations. These functions may be complex valued functions of the real variables \( s \) and \( t \).

Integral equations occur naturally in many fields of mechanics and mathematical physics. They also arise as representation formulas for the solutions of differential equations. Indeed,

A differential equation can be replaced by an integral equation

Which incorporates its boundary conditions? As such, each solution of the integral equation automatically satisfies these boundary conditions. Integral equations also form one of
The most useful tools in many branches of pure analysis, such as the theories of functional analysis and stochastic processes. One can also consider integral equations in which the unknown function is dependent not only on one variable but on several variables such, for example, is the equation

\[ g(s) = f(s) + \int_{\Omega} k(s,t)g(t) \, dt \quad (4) \]

Where \( s \) and \( t \) are \( n \)-dimensional vectors and \( \Omega \) is a region of an \( N \)-dimensional space. Similarly, one can also consider systems of integral equations with several unknown functions.

An integral equation is called linear if only linear operations are performed in it upon the unknown function. The equations (1) and (2) are linear, while (3) is nonlinear. In fact, the equations (1) and (2) can be written as

\[ L[g(s)] = f(s) \quad (5) \]

Where \( L \) is the appropriate integral operator. Then, for any constants \( c_1 \) and \( c_2 \), we have

\[ L[c_1g_1 + c_2g_2] = c_1L[g_1] + c_2L[g_2] \quad (6) \]

This is the general criterion for a linear operator in this book, we shall deal only with linear integral equations. The most general type of linear integral equation is of the form

\[ h(s)g(s) = f(s) + \lambda \int_{a}^{b} k(s,t)g(t) \, dt \]

Where the upper limit may be either variable or fixed; the function \( f, h \) and \( k \) are known functions, while \( g \) is to be determined; \( \lambda \) is a nonzero, real complex, parameter. The function \( k(s,t) \) is called the kernel.

The following special cases of equation (7) are of main interest.
1.1.1 FREDHOLM INTEGRAL EQUATIONS

In all fredholm integral equations, the upper limit of integration b Say is fixed

i) In the fredholm integral equation of the first kind, \( h(s) = 0 \)

Thus,

\[
f(s) + \lambda \int_{a}^{b} k(s, t)g(t) \, dt = 0 \tag{8}
\]

ii) In the fredholm integral equation of the second kind, \( h(s) = 1 \)

\[
g(s) = f(s) + \lambda \int_{a}^{b} k(s, t)g(t) \, dt \tag{8}
\]

iii) The homogeneous fredholm integral equation of the second kind is the special case of (ii) above. In this case, \( f(s) = 0 \)

\[
g(s) = \lambda \int_{a}^{b} k(s, t)g(t) \, dt \tag{8}
\]

1.1.2 VOLterra EQUATIONS

Volterra equations of the first, homogeneous, and second kinds are defined precisely as above except that \( b=s \) is the variable upper limit of integration.

Equation (7) itself is called an integer equation of the third kind
1.1.3 SINGULAR INTEGRAL EQUATIONS

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is called singular. For example

\[ g(s) = f(s) + \lambda \int_{-\infty}^{\infty} (\exp[-s-t])g(t) \, dt \]  \hspace{1cm} (11)

And

\[ f(s) = \int_{0}^{s} \frac{1}{(s-t)^\alpha} g(t) \, dt , \hspace{0.5cm} 0 < \alpha < 1 \]  \hspace{1cm} (12)

Are singular integral equations.

1.2 REGULARITY CONDITIONS

We shall be mainly concerned with functions which are either Continuous, or integrable or square integrable. In the subsequent analysis it will be pointed out that regularity conditions are expected of the functions involved. The notion of Lebesgue-integration is essential to modern mathematical analysis. When an integral sign is used, the lebesgue integral is understood. Fortunately if a function is Riemann-integrable it's also lebesgue-integrable there are functions that are Lebesgue-integrable but not Riemann-integrable, but we shall not encounter them in this book. Incidentally, by square-integrable, view [1]
Function $g(t)$, we mean that

$$\int_{a}^{b} |g(t)|^2 \, dt < \infty \quad (1)$$

This is called an $L_2$-function. The regularity conditions on the kernel $K(s,t)$ as a function of two variables are similar. It is an $L_2$-function if:

a) For each set of values of $s, t$ in the square $a \leq s \leq b, a \leq t \leq b$

$$\iint_{a}^{b} |k(s,t)|^2 \, ds \, dt < \infty \quad (2)$$

b) For each value of $s$ in $a \leq s \leq b$

$$\int_{a}^{b} |k(s,t)|^2 \, dt < \infty \quad (3)$$

c) For each value of $t$ in $a \leq t \leq b$

$$\int_{a}^{b} |k(s,t)|^2 \, ds < \infty \quad (4)$$

1.3 SPECIAL KINDS OF KERNELS

1.3.1 SEPARABLE OR DEGENERATE KERNEL

A kernel $k(s,t)$ is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of $s$ only and a function of $t$ only, i.e.,
The functions \( a_i(s) \) can be assumed to be linearly independent; otherwise the number of terms in relation (1) can be reduced.

(By linearly independent its meant that, if \( c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0 \) where \( c_i \) are arbitrary constants then \( c_1 = c_2 = \cdots = c_n = 0 \))

### 1.3.2 SYMMETRIC KERNEL

A complex valued function \( K(s, t) \) is called symmetric (or Hermitian) if

\[
K(s, t) = k^*(t, s)
\]

Where the asterisk denotes the complex conjugate for areal kernel this coincides with definition

\[
K(s, t) = k(t, s)
\]

### 1.4 EIGENVALUES AND EIGENFUNCTIONS

If we write the homogeneous fredholm equation as

\[
\int_{a}^{b} k(s, t) g(t) \, dt = \mu g(s) \quad \mu = 1/\lambda
\]

We have the classical eigenvalue or characteristic value problem

\( \mu \) is the eigenvalue and \( g(t) \) is the corresponding eigenfunction

Or characteristic function since the linear integral equations is

Studied in the form (1.1.10) it is \( \lambda \) and not \( 1/\lambda \) which is called the eigenvalue, view [2]
1.5 CONVOLUTION INTEGRAL

Many interesting problems of mechanics and physics lead to an integral equation in which the kernel $k(s, t)$ is a function of the difference $(s - t)$ only:

$$k(s, t) = k(s - t) \quad (1)$$

Where $k$ is a certain function of one variable. The integral equation

$$g(s) = f(s) + \lambda \int_{a}^{s} k(s - t)g(t) \, dt \quad (2)$$

and the corresponding fredholm equations are called integral equations of the convolution type.

The function defined by the integral

$$\int_{0}^{s} k(s - t)g(t) \, dt = \int_{0}^{s} k(t)g(s - t) \, dt$$

Is called the convolution or the faltung of the two functions $k$ and $g$ the integrals occurring in (3) are called the convolution integrals

The convolution defined by the relation (3) is a special case of the standard convolution

$$\int_{-\infty}^{\infty} k(s - t)g(t) \, dt = \int_{-\infty}^{\infty} k(t)g(s - t) \, dt$$

The integrals in (3) are obtained from those in (4) by taking

$$k(t) = g(t) = 0 \quad \text{for} \ t < 0 \quad t > s$$
1.6 THE INNER OR SCALAR PRODUCT OF TWO FUNCTIONS

The inner or scalar product \( (\phi, \psi) \) of two complex \( L_2 \) function \( \phi \) and \( \psi \) of a real variable \( s, \ a \leq s \leq b \) is defined as

\[
(\phi, \psi) = \int_{a}^{b} \phi(t)\psi^*(t) \, dt
\]

Two functions are called orthogonal if their inner product is zero

That is \( \phi \) and \( \psi \) are orthogonal if \( (\phi, \psi) = 0 \), the norm of a function \( \phi \) is given by the relation

\[
\|\phi(t)\| = \left[ \int_{a}^{b} \phi(t)\phi^*(t) \, dt \right]^{1/2} = \left[ \int_{a}^{b} |\phi(t)|^2 \, dt \right]^{1/2}
\]

A function \( \phi(t) \) is called normalized if \( \|\phi\| = 1 \) it is obvious that a non-null function (whose norm is not zero) can always be normalized by dividing it by its norm.

We shall have a great deal more to say about these ideas in [1].

For the time being, we shall content ourselves with mentioning the SCHWARZ and MINKOWSKI inequalities

\[
|\phi, \psi| \leq \|\phi\|\|\psi\| \quad (3)
\]

And

\[
\|\phi + \psi\| \leq \|\phi\| + \|\psi\| \quad (4)
\]

Respectively, view [1]
1.7 NOTATION

For Fredholm integral equations, it will be assumed that the range of integration is $a$ to $b$, unless the contrary is explicitly stated.

The quantities $a$ and $b$ will be omitted from the integral sign in the sequel.

1.8 REDUCTION TO A SYSTEM OF ALGEBRAIC SEPARABLE KERNELS

We have defined a degenerate or a separable kernel $K(s, t)$ as

$$k(s, t) = \sum_{i=1}^{n} a_i(s)b_i(t), \quad (1)$$

Where the functions $a_1(s)$, $a_n(s)$ and the functions $b_1(t)$, ..., $b(t)$ are linearly independent. With such a kernel, the Fredholm integral equation of the second kind,

$$g(s) = f(s) + \lambda \int k(s, t)g(t) \, dt \quad (2)$$

Becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int b_i(t)g(t) \, dt \quad (3)$$

It emerges that the technique of solving this equation is essentially dependent on the choice of the complex parameter $\lambda$. And on the definition of

$$c_i = \int b_i(t)g(t) \, dt \quad (4)$$
The quantities \( c_i \) are constants, although hitherto unknown. Substituting (4) in (3) gives

\[
g(s) = f(s) + \lambda \sum_{i=1}^{n} c_i a_i(s) \tag{5}
\]

And the problem reduces to finding the quantities \( c_i \). To this end, we put the value of \( g(s) \) as given by (5) in (3) and get

\[
\sum_{i=1}^{n} a_i(s) \{ c_i - \int b_i(t) [f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t)] \, dt \} = 0 \tag{6}
\]

But the functions \( a_i(s) \) are linearly independent; therefore,

\[
c_i - \int b_i(t) \left[ f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right] \, dt = 0 \quad i = 1, \ldots, n \tag{7}
\]

Using the simplified notation

\[
\int b_i(t) f(t) \, dt = f_i, \quad \int b_i(t) a_k(t) \, dt = a_{ik} \tag{8}
\]

Where \( f_i \) and \( a_{ik} \) are known constants, equation (7) becomes

\[
c_i - \lambda \sum_{k=1}^{n} c_k a_{ik}(t) = f_i, \quad i = 1, \ldots, n \tag{9}
\]

That is, a system of \( n \) algebraic equations for the unknowns \( c_i \). The determinant \( D(\lambda) \) of this system is

\[
\begin{vmatrix}
1 - \lambda a_{11} & -\lambda a_{12} & \ldots & -\lambda a_{1n} \\
-\lambda a_{21} & 1 - \lambda a_{22} & \ldots & -\lambda a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-\lambda a_{n1} & -\lambda a_{n2} & \ldots & 1 - \lambda a_{nn}
\end{vmatrix} \tag{10}
\]
Which is a polynomial in \( \lambda \) of degree at most \( n \). Moreover, it is not identically zero, since, when \( \lambda = 0 \), it reduces to unity.

For all values of \( \lambda \). For which \( D(\lambda) \neq 0 \), the algebraic system (9), and thereby the integral equation (2), has a unique solution. On the other hand, for all values of \( \lambda \). For which \( D(\lambda) \) becomes equal to zero, the algebraic system (9), and with it the integral equation (2), either is insoluble or has an infinite number of solutions. Setting \( \lambda = 1/\mu \) in Equation (9), we have the eigenvalue problem of matrix theory. The eigenvalue are given by the polynomial \( D(\lambda) = 0 \). They are also the eigenvalues of our integral equation.

Note that we have considered only the integral equation of the second kind, where alone this method is applicable.

This method is illustrated with the following examples

1.9 EXAMPLES, view [5]

1.9.1 Example 1 Solve the Fredholm integral equation of the second kind

\[
g(s) = s + \lambda \int_0^1 (st^2 + s^2t) g(t) \, dt ,
\]

(1)

The kernel \( k(s, t) = st^2 + s^2t \) is separable and we can set

\[
c_1 = \int_0^1 t^2 g(t) \, dt , \quad c_2 = \int_0^1 t g(t) \, dt .
\]

Equation (1) becomes

\[
g(S) = s + \lambda c_1 s + \lambda c_2 s^2 ,
\]

(2)

Which we substitute in (1) to obtain the algebraic equations
\( c_1 = \frac{1}{4} + \frac{1}{4} \lambda c_1 + \frac{1}{5} \lambda c_2 \), \\
\( c_2 = \frac{1}{3} + \frac{1}{3} \lambda c_1 + \frac{1}{4} \lambda c_2 \) . \hspace{2cm} (3)

The solution of these equations is readily obtained as

\[ c_1 = \frac{60 + \lambda}{240 - 120 \lambda - \lambda^2} , \]

\[ c_2 = \frac{80}{(240 - 120 \lambda - \lambda^2)} \] \hspace{2cm} (4)

From (2) and (4), we have the solution

\[ g(s) = \frac{[(240 - 60 \lambda)s + 80 \lambda s^2]}{240 - 120 \lambda - \lambda^2} \]

1.9.2 Example 2 Solve the integral equation

\[ g(s) = f(s) + \lambda \int_0^1 (s + t) g(t) dt . \hspace{2cm} (6) \]

And find the eigenvalue.

Here, \( a_1(s) = s, \ a_2(s) = 1, \ b_1(t) = 1, \ b_2(t) = t \)

\[ a_{11} = \int_0^1 t dt = \frac{1}{2} \hspace{1cm} , \hspace{1cm} a_{12} = \int_0^1 dt = 1 \]

\[ a_{21} = \int_0^1 t^2 dt = \frac{1}{3} \hspace{1cm} , \hspace{1cm} a_{22} = \int_0^1 tf(t) dt = \frac{1}{2} \]
\[ f_1 = \int_0^1 f(t)dt, \quad f_2 = \int_0^1 t f(t)dt. \]

Substituting these values in (2.1.9), we have the algebraic system

\[ (1 - \frac{1}{2}\lambda) c_1 - \lambda c_2 = f_1, \quad -\frac{1}{3} \lambda c_1 + (1 - \frac{1}{2}\lambda c_2) = f_2 \]

The determinant \( D(\lambda) = 0 \) gives \( \lambda + 12\lambda - 12 = 0 \)

Thus, the eigenvalue are \( \lambda_1 = (-6 + 4\sqrt{3}) \), \( \lambda_2 = (-6 - 4\sqrt{3}) \).

For these two values of \( \lambda \), the homogeneous equation has a nontrivial solution, while the integral equation (6) is, in general, not soluble. When \( A, \) differs from these values, the solution of the above algebraic system is

\[ c_1 = \frac{[-12f_1 + \lambda(6f_1 - 12f_2)]}{\lambda^2 + 12\lambda - 12}, \]

\[ c_2 = \frac{[-12f_2 - \lambda(4f_1 - 6f_2)]}{\lambda^2 + 12\lambda - 12}. \]

Using the relation (2.1.5), there results the solution

\[ g(s) = f(s) + \lambda \int_0^1 \frac{[6(\lambda - 2)(s + t) - 12\lambda st - 4\lambda]}{[\lambda^2 + 12\lambda - 12]} \quad f(t) \quad d(t), \quad (7) \]

The function \( \Gamma(s, t; \lambda) \)

\[ \Gamma(s, t; \lambda) = \frac{[6(\lambda - 2)(s + t) - 12\lambda st - 4\lambda]}{(\lambda^2 + \lambda - 12)}, \quad (8) \]
Is called the resolvent kernel. We have therefore succeeded in inverting the integral equation because the right-hand side of the above formula is a known quantity.

**1.9.3 Example 3** Invert the integral equation

\[
g(s) = f(s) + \lambda \int_0^{2\pi} (\sin s \cos t) g(t) dt \quad (9)
\]

As in the previous examples, we set

\[
c = \int_0^{2\pi} (\cos t) g(t) dt
\]

To obtain

\[
g(s) = f(s) + \lambda c \sin s \quad (9)
\]

Multiply both sides of this equation by \(\cos(s)\) and integrate from 0 to \(2\pi\). This gives

\[
c = \int_0^{2\pi} (\cos t) g(t) dt \quad (11)
\]
From (10) and (11), we have the required formula:

\[ f(s) + \lambda \int_0^{2\pi} (\sin s \cos t)f(t)\,dt \quad (12) \]

**1.9.4 Example 4** Find the resolvent kernel for the integral equation

\[ g(s) = f(s) + \lambda \int_{-1}^{1} (st + s^2t^2)g(t)\,dt \quad (13) \]

For this equation,

\[ a_1(s) = s, \quad a_2(s) = s^2, \quad b_1(t) = t, \quad b_2(t) = t^2 \]

\[ a_{11}(s) = \frac{2}{3}, \quad a_{12}(s) = a_{21}(s) = 0, \quad a_{22}(s) = \frac{2}{5} \]

\[ f_1 = \int_{-1}^{1} tf(t)\,dt, \quad f_2 = \int_{-1}^{1} t^2 f(t)\,dt \]

Therefore, the corresponding algebraic system is

\[ c_1(1 - 2/3\lambda) = f_1, \quad c_2(1 - 2/5\lambda) = f_2 \quad (14) \]
Substituting the values of $c_1$ and $c_2$ as obtained from (14) in (2.1.9) yields the solution

$$g(s) = f(s) + \lambda \int_{-1}^{1} \left[ \left( \frac{st}{1 - \frac{2}{3} \lambda} \right) + \left( \frac{s^2 t^2}{1 - \frac{2}{5} \lambda} \right) \right] f(t) \, dt \quad (15)$$

Thus, the resolvent kernel is

$$\Gamma(s, t; \lambda) = \left( \frac{st}{1 - \frac{2}{3} \lambda} \right) + \left( \frac{s^2 t^2}{1 - \frac{2}{5} \lambda} \right) \quad (16)$$

We shall now give examples of homogeneous integral equations.

**1.9.5 Example 5** Solve the homogeneous Fredholm integral equation

$$g(s) = \lambda \int_{0}^{1} e^{s} e^{t} g(t) \, dt \quad (17)$$

Define

$$c = \int_{0}^{1} e^{t} g(t) \, dt$$

So that (17) becomes

$$g(s) = \lambda ce^{s} \quad (18)$$
Put this value of \( g(s) \) in (17) and get

\[
\lambda ce^s = \lambda e^s \int_0^1 e^t [\lambda ce^t] = \frac{1}{2} \lambda^2 e^s c(e^2 - 1)
\]

Or

\[
\lambda c\{2 - \lambda (e^2 - 1)\} = 0
\]

If \( c = 0 \) or \( \lambda = 0 \), then we find that \( g \equiv 0 \). Assume that neither

\( c = 0 \) nor \( \lambda = 0 \) then, we have the eigenvalue

\[
\lambda = \frac{2}{e^2 - 1}
\]

Only for this value of \( \lambda \) does there exist a nontrivial solution of the integral equation (17). This solution is found from (18) to be

\[
g(s) = \left[\frac{2c}{(e^2 - 1)}\right] e^s \quad (19)
\]

Thus, to the eigenvalue \( (e^2 - 1) \) there corresponds the eigenfunction \( e^s \).

**1.9.6 Example 6** Find the eigenvalue and eigenfunction of the homogeneous integral equation

\[
g(s) = \lambda \int_1^2 \left[st + \left(\frac{1}{st}\right)\right] g(t) dt \quad (20)
\]
Comparing this with (2.1.3), we have

\[ a_1(s) = s, \quad a_2(s) = 1/s, \quad b_1(t) = t, \quad b_2(t) = 1/t \]

\[ a_{11} = \frac{7}{3}, \quad a_{12} = a_{21} = 1, \quad a_{22} = 1/2 \]

The formula (2.1.9) then yields the following homogeneous equations:

\[
\begin{align*}
(1 - \frac{7}{3}\lambda)c_1 - \lambda c_2 &= 0, \\
-\lambda c_1 + (1 - \frac{1}{2}\lambda)c_2 &= 0
\end{align*}
\]

Which have a nontrivial solution only if the determinant

\[
D(\lambda) = \begin{vmatrix}
1 - \frac{3}{7}\lambda & -\lambda \\
-\lambda & 1 - \lambda
\end{vmatrix} = 1 - \left(\frac{17}{6}\right)\lambda + \left(\frac{1}{6}\right)^2,
\]

vanishes. Therefore, the required eigenvalue are

\[ \lambda_1 = \frac{1}{2}\left(17 + \sqrt{265}\right) \approx 16.6394 \]

\[ \lambda_2 = \frac{1}{2}\left(17 - \sqrt{265}\right) \approx 0.3606 \]  

The solution corresponding to \( \lambda_1 \) is \( C_2 \approx -2.2732C_1 \), while that corresponding to \( \lambda_2 \) is \( C_2 \approx 0.3499C_1 \). The eigenfunctions of the given integral equation are found by substituting in (2.1.5):

\[
\begin{align*}
g_1(s) &= 16.639C_1\left[ s - 2.2732\left(\frac{1}{s}\right) \right], \\
g_2(s) &= .3606C_1\left[ s + 0.4339\left(\frac{1}{s}\right) \right]
\end{align*}
\]

Where \( C_1 \) and \( C_1' \) are two undetermined constants.
1.10 FREDHOLM ALTERNATIVE

In the previous sections, we have found that, if the kernel is separable, the problem of solving an integral equation of the second kind reduces to that of solving an algebraic system of equations. Unfortunately, integral equations with degenerate kernels do not occur frequently in practice. But since they are easily treated and, furthermore, the results derived for such equations lead to a better understanding of integral equations of more general types, it is worthwhile to study them. Last, but not least, any reasonably well-behaved kernel can be written as an infinite series of degenerate kernels.

When an integral equation cannot be solved in closed form, then recourse has to be taken to approximate methods. But these approximate methods can be applied with confidence only if the existence of the solution is assured in advance. The Fredholm theorems explained in this chapter provide such an assurance. The basic theorems of the general theory of integral equations, which were first presented by Fredholm, correspond to the basic theorems of linear algebraic systems. Fredholm's classical theory shall be presented in Chapter 2 for general kernels. Here, we shall deal with degenerate kernels and borrow the results of linear algebra.

In Section 2.1, we have found that the solution of the present problem rests on the investigation of the determinant (2.1.10) of the coefficients of the algebraic system (2.1.9).
If $D(A) \neq 0$ then that system has only one solution, given by Cramer's rule

$$c_i = \frac{D_1f_1 + D_2f_2 + \cdots + D_nf_n}{D(\lambda)} \quad i = 1,2,\ldots,n$$

(1)

Where $D_{hi}$ denotes the cofactor of the $(h, i)$ th element of the determinant (2.1.10). Consequently, the integral equation (2.1.2) has the unique solution (2.1.5), which, in view of (1), becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} \frac{D_1f_1 + D_2f_2 + \cdots + D_nf_n}{D(\lambda)} a_i(s)$$

(2)

While the corresponding homogeneous equation

$$g(s) = \lambda \int k(s,t)g(t)dt$$

(3)

has only the trivial solution $g(s) = 0$.

Substituting for $f_i$ from (2.1.8) in (2), we can write the solution $g(s)$ as

$$g(s) = f(s) + \left[ \frac{\lambda}{D(\lambda)} \right] \int \left\{ \sum_{i=1}^{n} [D_1b_1 + D_2b_2 + \cdots + D_nb_n]a_i(s) \right\} f(t)dt$$

(4)

Now consider the determinant of $(n + 1)$th order

$$D(s,t; \lambda) = -\begin{vmatrix} 0 & a_1(s) & \cdots & a_n(s) \\ b_1(t) & 1 - \lambda a_{11} & \cdots & -\lambda a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n(t) & -\lambda a_{n1} & \cdots & 1 - \lambda a_{nn} \end{vmatrix}$$

(5)

By developing it by the elements of the first row and the corresponding minors by the elements of the first column, we find that the expression in the brackets in equation (4) is $D(s,t; \lambda)$ with the definition
\[ \Gamma(s, t; \lambda) = D(s, t; \lambda)/D(\lambda) \quad (6) \]

Equation (4) takes the simple form

\[ g(s) = f(s) + \lambda \int \Gamma(s, t; \lambda)f(t)dt \quad (7) \]

The function \( \Gamma(s, t; \lambda) \) is the resolvent (or reciprocal) kernel we have already encountered in Examples 2 and 4 in the previous section. We shall see in [4] that the formula (6) has many important consequences. For the time being, we content ourselves with the observation that the only possible singular points of \( \Gamma(s, t; \lambda) \) in the \( .1 \)-plane are the roots of the \( (\lambda) = 0 \), i.e., the eigenvalues of the kernel \( k(s, t) \).

The above discussion leads to the following basic Fredholm theorem.

Fredholm Theorem. The inhomogeneous Fredholm integral equation (2.1.2) with a separable kernel has one and only one solution, given by formula (7). The resolvent kernel \( \Gamma(s, t; \lambda) \) coincides with the quotient (6) of two polynomials.

If \( D(\lambda) = 0 \), then the inhomogeneous equation (2.1.2) has no solution in general, because an algebraic system with vanishing determinant can be solved only for some particular values of the quantities \( A \). To discuss this case, we write the algebraic system (2.1.9) as

\[ (I - \lambda A)c = f \quad (8) \]

Where \( I \) is the unit (or identity) matrix of order \( n \) and \( A \) is the matrix \( (a_{ij}) \). Now, when \( D(\lambda) = 0 \), we observe that for each nontrivial solution of the homogeneous algebraic system

\[ (I - \lambda A)c = 0 \quad (9) \]
There corresponds a nontrivial solution (an eigenfunction) of the homogeneous integral equation (3). Furthermore, if A coincides with a certain eigenvalue $\lambda_0$ for which the determinant $D(\lambda_0) = |I - \lambda_0 A|$ has the rank $p$, $1 \leq p \leq n$, then there are $r = n - p$ linearly independent solutions of the algebraic system (9); $r$ is called the index of the eigenvalue $\lambda_0$. The same holds for the homogeneous integral equation (3). Let us denote these $r$ linearly independent solutions as $g_{01}(s), g_{02}(s), \ldots, g_{0r}(s)$, and let us also assume that they have been normalized. Then, to each eigenvalue $\lambda_0$ of index $r = n - p$, there corresponds a solution $g_0(s)$ of the homogeneous integral equation (3) of the form

$$g_0(s) = \sum_{k=1}^{r} \alpha_k g_{0k}(s)$$

Where $\alpha_k$ are arbitrary constants?

Let $m$ be the multiplicity of the eigenvalue $\lambda_0$, i.e., $D(\lambda) = 0$ has $m$ equal roots $\lambda_0$. Then, we infer from the theory of linear algebra that, by using the elementary transformations on the determinant $|I - \lambda A|$, we shall have at most $m + 1$ identical rows and this maximum is achieved only if $A$ is symmetric. This means that the rank $p$ of $D (\lambda_0)$ is greater than or equal to $n - m$. Thus,

$$r = n - p \leq n - (n - m) = m,$$

And the equality holds only when $a_{ij} = a_{ji}$.

Thus we have proved the theorem of Fredholm that, if $\lambda = \lambda_0$ is a root of multiplicity $m \geq 1$ of the equation $D(\lambda) = 0$, then the homogeneous integral equation (3) has $r$ linearly independent solutions; $r$ is the index of the eigenvalue such that $1 \leq r \leq m$. 
The numbers \( r \) and \( m \) are also called the geometric multiplicity and algebraic multiplicity of \( \lambda_0 \), respectively. From the above result, it follows that the algebraic multiplicity of an eigenvalue must be greater than or equal to its geometric multiplicity.

To study the case when the inhomogeneous Fredholm integral equation (2.1.2) has solutions even when \( D(\lambda) = 0 \), we need to define and study the transpose of the equation (2.1.2). The integral equation:

\[
\psi(s) = f(s) + \lambda \int K(t,s) \psi(t) dt \quad (10)
\]

is called the transpose (or adjoint) of the equation (2.1.2). Observe that the relation between (2.1.2) and its transpose (10) is symmetric, since (2.1.2) is the transpose of (10).

If the separable kernel \( k(s,t) \) has the expansion (2.1.1), then the kernel \( K(t,s) \) of the transposed equation has the expansion

\[
K(t,s) = \sum_{i=1}^{n} a_i(t)b_i(s) \quad (11)
\]

Proceeding as in Section 2.1, we end up with the algebraic system

\[
(I - \lambda A^T)c = f \quad (12)
\]

Where \( AT \) stands for the transpose of \( A \) and where \( c_i \) and \( f_i \) are now defined by the relations

\[
c_i = \int a_i(t)\psi(t)dt, \quad f_i = \int a_i(t)f(t)dt \quad (13)
\]
The interesting feature of the system (12) is that the determinant $D(\lambda)$ is the same function as (2.1.10) except that there has been an interchange of rows and columns in view of the interchange in the functions $a_i$ and $b_i$. Thus, the eigenvalue of the transposed integral equation are the same as those of the original equation. This means that the transposed equation (10) also possesses a unique solution whenever (2.1.2) does.

As regards the eigenfunction of the homogeneous system

$$(I - \lambda A^T)c = 0, \quad (14)$$

We know from linear algebra that these are different from the corresponding eigenfunction of the system (9). The same applies to the eigenfunction of the transposed integral equation. Since the index $r$ of $\lambda_0$ is the same in both these systems, the number of linearly independent eigenfunction is also $r$ for the transposed system. Let us denote them by $\psi_{01}, \psi_{02}, \ldots, \psi_{0r}$ and let us assume that they have been normalized.

Then, any solution $\psi_0(s)$ of the transposed homogeneous integral equation

$$\psi(s) = \lambda \int K(t, s) \psi(t) dt \quad (15)$$

Corresponding to the eigenvalue $\lambda_0$ is of the form

$$\psi(s) = \sum \beta_i \psi_{0i}(s)$$

Where $\beta_i$ are arbitrary constants
We prove in passing that eigenfunctions \( g(s) \) and \( \psi(s) \) corresponding to distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively, of the homogeneous integral equation (3) and its transpose (15) are orthogonal. In fact, we have

\[
\begin{align*}
g(s) &= \lambda_1 \int k(s,t)g(t)dt \\
\psi(s) &= \lambda_2 \int k(s,t)\psi(t)dt
\end{align*}
\]

Multiplying both sides of the first equation by \( \lambda_2 \psi(s) \) and those of the second equation by \( \lambda_1 g(s) \), integrating, and then subtracting the resulting equations, we obtain

\[
(\lambda_2 - \lambda_1) \int_a^b g(s)\psi(s)ds
\]

But \( \lambda_2 \neq \lambda_1 \) and the result follows.

We are now ready to discuss the solution of the inhomogeneous Fredholm integral equation (2.1.2) for the case \( D(\lambda) = 0 \). In fact, we can prove that the necessary and sufficient condition for this equation to have a solution for \( \lambda = \lambda_0 \), a root of \( D(\lambda) = 0 \), is that \( f(s) \) be orthogonal to the \( r \) eigenfunctions \( \psi_{0l} \) of the transposed equation (15).

The necessary part of the proof follows from the fact that, if equation (2.1.2) for \( \lambda = \lambda_0 \), admits a certain solution \( g(s) \), then

\[
\begin{align*}
\int f(s)\psi_{0l}(s) \, ds &= \int g(s)\psi_{0l}(s) \, ds - \lambda_0 \int \psi_{0l}(s) \, ds \int k(s,t)g(t) \, dt \\
&= \int g(s)\psi_{0l}(s) \, ds - \lambda_0 \int g(t) \, dt \int k(s,t)\psi_{0l}(s) \, ds = 0
\end{align*}
\]
Because $\lambda_0$ and $\psi_{0i}(s)$ are eigenvalues and corresponding eigenfunctions of the transposed equation. To prove the sufficiency of this condition, we again appeal to linear algebra. In fact, the corresponding condition of orthogonality for the linear-algebraic system assures us that the inhomogeneous system (8) reduces to only $n - r$ independent equations. This means that the rank of the matrix $(I - \lambda A)$ is exactly $p = n - r$ and therefore the system (8) or (2.1.9) is soluble. Substituting this solution in (2.1.5), we have the solution to our integral equation.

Finally, the difference of any two solutions of (2.1.2) is a solution of the homogeneous equation (3). Hence, the most general solution of the inhomogeneous integral equation (2.1.2) has the form

$$g(s) = G(s) + \alpha_1 g_{01}(s) + \alpha_2 g_{02}(s) + ... + \alpha_r g_{0r}(s)$$

(16)

Where $G(s)$ is a suitable linear combination of the functions

$$a_1(s), a_2(s), ... , a_n(s)$$

We have thus proved the theorem that, if $\lambda = \lambda_0$ is a root of multiplicity $m \geq 1$ of the equation $D(\lambda) = 0$, then the inhomogeneous equation has a solution if and only if the given function $f(s)$ is orthogonal to all the eigenfunctions of the transposed equation.

The results of this section can be collected to establish the following theorem.

### 1.11 Fredholm Alternative Theorem

Either the integral equation

$$g(s) = f(s) + \lambda \int K(s, t)g(t)dt$$

(17)

With fixed $\lambda$ possesses one and only one solution $g(s)$ for arbitrary $g2 - functions f(s)$ and $K(s, t)$, in particular the solution $g = 0$ for $f = 0$; or the homogeneous equation
\[ g(s) = \lambda \int K(s,t) g(t) dt \quad (18) \]

Possesses a finite number \( r \) of linearly independent solutions \( g_{0i}, \ i = 1,2,\ldots, r \). In the first case, the transposed in homogeneous equation

\[ \psi(s) = f(s) + \lambda \int K(t,s) \psi(t) dt \quad (19) \]

Also possesses a unique solution. In the second case, the transposed homogeneous equation

\[ \psi(s) = \lambda \int K(t,s) \psi(t) dt \quad (20) \]

Also has \( r \) linearly independent solutions \( \psi_{0i}, \ i = 1,2,\ldots, r \); the in-homogeneous integral equation (7) has a solution if and only if the given function \( f(s) \) satisfies the \( r \) conditions

\[ (f,\psi_{0i}) = \int f(s) \psi_{0i}(s) ds = 0, \quad i = 1,2,\ldots, r \]

In this case, the solution of (17) is determined only up to an additive linear combination

\[ \sum_{i=1}^{r} c_i g_{0i} \cdot \]

The following examples illustrate the theorems of this section. , view [1]
1.12 EXAMPLES

1.12.1 Example 1  Show that the integral equation

\[ g(s) = f(s) + \frac{1}{\pi} \int_{0}^{2\pi} [\sin(s + t)]g(t)dt \]  \hspace{1cm} (1)\]

Possesses no solution for \( f(s) = s \), but that it possesses infinitely many solutions when \( f(s) = 1 \).

For this equation,

\[ K(s, t) = \sin s \cos t + \cos s \sin t, \]

\[ a_1(s) = \sin s, \quad a_2(s) = \cos s, \quad b_1(t) = \cos t, \quad b_2 = \sin t. \]

Therefore,

\[ a_{11} = \int_{0}^{2\pi} \sin t \cos t \, dt = 0 = a_{22} \]

\[ a_{12} = f \int_{0}^{2\pi} \cos^2 t \, dt = \pi = a_{21} \]

\[ D(\lambda) = \begin{vmatrix} 1 & -\lambda \pi \\ -\lambda \pi & 1 \end{vmatrix} = 1 - \lambda^2 \pi^2 \]  \hspace{1cm} (2)
The eigenvalues are $\lambda_1 = 1/\pi$, $\lambda_2 = -1/\pi$ and equation (1) contains $\lambda=1/\pi$. Therefore, we have to examine the eigenfunctions of the transposed equation (note that the kernel is symmetric).

\[
g(s) = \left(\frac{1}{\pi}\right) \int_0^{2\pi} \sin(s + t) g(t) dt \quad (3)
\]

Therefore, the eigenfunctions for $\lambda_1 = 1/\pi$ follow from the relation (2.1.5) and are given by the algebraic system corresponding to (3) is

\[
c_1 - \lambda \pi c_2 = 0, \quad -\lambda \pi c_1 + c_2 = 0
\]

which gives

\[
c_1 = c_2 \quad \text{for} \quad \lambda_1 = 1/\pi; \quad c_1 = -c_2 \quad \text{for} \quad \lambda_2 = -1/\pi
\]

\[
g(s) = c (\sin(s) + \cos(s)) \quad (4)
\]

Since

\[
\int_0^{2\pi} (s \sin(s) + s \cos(s)) ds = -2\pi \neq 0,
\]

while

\[
\int_0^{2\pi} (s \sin(s) + s \cos(s)) ds = 0
\]

we have proved the result.

1.2 Example 2 Solve the integral equation

\[
g(s) = f(s) + \lambda \int_0^1 (1 - 3st) g(t) dt \quad (5)
\]

The algebraic system (2.1.9) for this equation is
\[(1 - \lambda)c_1 + \frac{3}{2}\lambda c_2 = f_1, \quad -\frac{1}{2}\lambda c_1 + (1 + \lambda)c_2 = f_2 \quad (6)\]

Therefore, the inhomogeneous equation (5) will have a unique solution if and only if \(\lambda = \pm 2\). Then the homogeneous equation

\[g(s) = \lambda \int_0^1 (1 - 3st) g(t) dt \quad (8)\]

has only the trivial solution.

Let us now consider the case when \(A\) is equal to one of the eigenvalues and examine the eigenfunctions of the transposed homogeneous equation

\[g(s) = \lambda \int_0^1 (1 - 3st) g(t) dt \quad (9)\]

For \(\lambda = + 2\), the algebraic system (6) gives \(c_1 = 3c_2\). Then, (2.1.5) gives the eigenfunction

\[g(s) = c(1 - s) \quad (10)\]

where \(C\) is an arbitrary constant. Similarly, for \(A = -2\), the corresponding eigenfunction is

\[g(s) = C(1 - 3s) \quad (11)\]

It follows from the above analysis that the integral equation

\[g(s) = f(s) + 2 \int_0^1 (1 - 3st) g(t) dt\]
Will have a solution if \( f(s) \) satisfies the condition

\[
\int_0^1 (1 - s) f(s) \, ds = 0
\]

while the integral equation

\[
g(s) = f(s) - 2 \int_0^1 (1 - 3st) g(t) \, dt
\]

will have a solution if the following holds:

\[
\int_0^1 (1 - 3s) f(s) \, ds = 0
\]

3 AN APPROXIMATE METHOD

The method of this chapter is useful in finding approximate solutions of certain integral equations. We illustrate it by the following example:

\[
g(s) = e^s - s - \int_0^1 (e^{st} - 1) g(t) \, dt \tag{1}
\]

Let us approximate the kernel by the sum of the first three terms in its Taylor series:

\[
K(t, s) = s (e^{st} - 1) = s^2 t + \frac{1}{2} s^3 t^2 + \frac{1}{6} s^4 t^3 \tag{2}
\]

that is, by a separable kernel. Then the integral equation takes the form

\[
g(s) = e^s - s - \int_0^1 s^2 t + \frac{1}{2} s^3 t^2 + \frac{1}{6} s^4 t^3 \tag{3}
\]

Since the kernel is separable, we require the solution in the form

\[
g(s) = e^s - s + c_1 s^2 + c_2 s^3 + c_3 s^4 \tag{4}
\]
Following the method of this chapter, we find that the constants \( c_1, c_2, c_3 \) satisfy the following algebraic system:

\[
\begin{align*}
(5/4) c_1 + (1/5) c_2 + (1/6) c_3 &= -2/3 \\
(1/5) c_1 + (13/6) c_2 + (1/7) c_3 &= (9/4) - e (5) \\
(1/6) c_1 + (1/7) c_2 + (49/8) c_3 &= 2e - (29/5)
\end{align*}
\]

Which solution is

\[
c_1 = -0.5010, \quad c_2 = -0.1671, \quad c_3 = -0.0422. \quad (6)
\]

Substituting these values in (4), we have the solution

\[
g (s) = e^s - s - 0.5010s^2 - 0.1671s^3 - 0.0423s^4. \quad (7)
\]

Now the exact solution of the integral equation (1) is

\[
g (s) = 1 \quad (8)
\]

Using the approximate solution for \( s = 0, s = 0.5, \) and \( s = 1, \) the value of \( g (s) \) from (7) is

\[
g (0) = 1.0000, \quad g(0.5) = 1.0000, \quad g(1) = 1.0080, \quad (9)
\]

Which agrees with (8) rather closely.
2.1 ITERATIVE SCHEME

Ordinary first-order differential equations can be solved by the well-known Picard of successive approximations. An iterative scheme based on the same principle is also available for linear integral equations of the second kind:

\[ g(s) = f(s) + \lambda \int k(s, t) g(t) dt \quad (1) \]

In this chapter, we present this method. We assume the functions \( f(s) \) and \( k(s, t) \) to be \( \mathcal{L}_2 \)-function as defined in Chapter 1 as a zero-order approximation to the desired function \( g(s) \), the solution \( g_0(s) \)

\[ g_0(s) = f(s) \quad (2) \]

Is taken. This is substituted into the right side of equation (1) to give the first-order approximation

\[ g_1(s) = f(s) + \lambda \int k(s, t) g_0(t) \, dt \quad (3) \]

This function, when substituted into (1), yields the second approximation. This process is then repeated; the (n+1) th approximation is obtained by substituted the nth approximation in the right side of (1) there results the recurrence relation

\[ g_{n+1}(s) = f(s) + \lambda \int k(s, t) g_n(t) \, dt \quad (4) \]

If \( g_n(s) \) tends uniformly to a limit as \( n \to \infty \), then this limit is the required solution. To study such a limit, let us examine the iterative procedure (4) in detail. The first –and second –order approximations are

\[ g_1(s) = f(s) + \lambda \int k(s, t) f(t) \, dt \quad (5) \]
\[ g_2(s) = f(s) + \lambda \int k(s, t) f(t) \, dt + \lambda^2 \int k(s, t) \left[ \int k(t, x) f(x) \, dx \right] dt \quad (6) \]

This formula can be simplified by setting
\[ k_2(s, t) = \int k(s, x) k(x, t) \, dx \quad (7) \]

And by changing the order of integration. The result is
\[ g_2(s) = f(s) + \lambda \int k(s, t) f(t) \, dt + \lambda^2 \int k_2(s, t) f(t) \, dt \quad (8) \]

Similarly,
\[ g_3(s) = f(s) + \lambda \int k(s, t) f(t) \, dt + \lambda^2 \int k_2(s, t) f(t) \, dt + \lambda^3 \int k_3(s, t) f(t) \, dt \quad (9) \]

Where
\[ k_3(s, t) = \int k(s, x) k_2(x, t) \, dx \quad (10) \]

By continuing this process, and denoting
\[ k_m(s, t) = \int k(s, x) k_{m-1}(x, t) \, dx \quad (11) \]

We get the \((n+1)\) th approximate solution of integral equation (1) as
\[ g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int k_m(s, t) f(t) \, dt \quad (12) \]

We call the expression \(k_m(s, t)\) the \(m\)th iterate, where \(k_1(s, t) = k(s, t)\)

Passing to the limit as \(n \to \infty\), we obtain the so-called Neumann series
\[ g(s) = \lim_{n \to \infty} g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int k_m(s, t) f(t) \, dt \quad (13) \]
It remains to determine conditions under which this convergence is achieved. For this purpose, we attend to the partial sum (12) and apply the Schwarz inequality (1.6.3) to the general term of this sum. This gives

$$\left| \int k_m(s, t) f(t) \, dt \right|^2 \leq \left( \int |k_m(s, t)|^2 \, dt \right) \int |f(t)|^2 \, dt \quad (14)$$

Let D be the norm of f,

$$D^2 = \int |f(t)|^2 \, dt, \quad (15)$$

And let $C_m^2$ denote the upper bound of the integral

$$\int |k_m(s, t)|^2 \, dt$$

Hence, the inequality (14) becomes

$$\left| \int k_m(s, t) f(t) \, dt \right|^2 \leq C_m^2 D^2 \quad (16)$$

The next step is to connect the estimate $C_m^2$ with the estimate $C_1^2$.

This is achieved by applying the Schwarz inequality to the relation (11):

$$|k_m(s, t)|^2 \leq \left( \int |k_{m-1}(s, x)|^2 \, dx \right) \int |k(x, t)|^2 \, dx$$

Which, when integrated with respect to t, yield

$$\int |k_m(s, t)|^2 \, dt \leq B^2 C_{m-1}^2 \quad (17)$$

Where

$$B^2 = \iint |k(x, t)|^2 \, dx \, dt \quad (18)$$

The inequality (17) sets up the recurrence relation

$$C_m^2 \leq B^{2m-2} C_1^2 \quad (19)$$

From (16) and (19), we have the inequality
\[
\left| \int k_m(s,t) f(t) \, dt \right|^2 \leq C_2^2 D^2 B^{2m-2} \quad (20)
\]

Therefore, the general term of the partial sum (12) has a magnitude less than the quantity \( DC_1 |\lambda| m B^{m-1} \), and it follows that the infinite series (13) converges faster than the geometric series with common ratio \( |\lambda| B \).

Hence, if \( |\lambda| B < 1 \) \quad (21)

The uniform convergence of this series is assured it will now be proved that, for given \( \lambda \), equation (1) has a unique solution. Suppose the contrary and let \( g_1(s) \) and \( g_2(s) \) be two solutions of equation (1):

\[
g_1(s) = f(s) + \lambda \int k(s,t) g_1(s) \, dt
\]

\[
g_2(s) = f(s) + \lambda \int k(s,t) g_1(s) \, dt
\]

By subtracting these equations and setting

\[
g_1(s) - g_2(s) = \phi(s) \text{ there results the homogeneous integral equation}
\]

\[
\phi(s) = \lambda \int k(s,t) \phi(t) \, dt
\]

Apply the Schwarz inequality to this equation and get

\[
|\phi(s)|^2 \leq |\lambda|^2 \int |k(s,t)|^2 \, dt \int |\phi(t)|^2 \, dt
\]

Which, when integrated with respect to \( s \), becomes

\[
\int |\phi(s)|^2 \, ds \leq |\lambda|^2 \int \int |k(s,t)|^2 \, ds \, dt \int |\phi(s)|^2 \, ds
\]

or

\[
(1 - |\lambda|^2 B^2) \int |\phi(s)|^2 \, ds \leq 0 \quad (22)
\]
In view of the inequality (21) and the nature of the function \( g_1(s) - g_2(s) = \phi(s) \) we readily conclude that \( \phi(s) = 0 \) i.e. \( g_1(s) = g_2(s) \) what is the estimate of the error for neglecting terms after the \( n \)th term in the Neumann series (13) this is found by writing this series as
\[
g(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int k_m(s, t) f(t) \, dt + R_n(s) \quad (23)
\]

Then, it follows from the above analysis that
\[
|R_n| \leq \frac{DC_1|\lambda|^{n+1}B^n}{(1 - |\lambda|B)} \quad (24)
\]

Finally, we can evaluate the resolvent kernel, as defined in the previous chapter, in terms of the iterated kernels \( k_m(s, t) \). I needed, by changing the order of integration and summation in the Neumann series (13) we obtain
\[
g(s) = f(s) + \lambda \int \left[ \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s, t) \right] f(t) \, dt
\]

Comparing this with (2.3.7)
\[
g(s) = f(s) + \lambda \int \Gamma(s, t; \lambda) f(t) \, dt \quad (25)
\]

We have
\[
\Gamma(s, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s, t) \quad (26)
\]

From the above discussion, we can infer (see section 3.5) that the series (26) is also convergent at least for \( |\lambda|B < 1 \). Hence the resolvent kernel is an analytic function of \( \lambda \) regular at least inside the circle \( |\lambda| < B^{-1} \) from the uniqueness of the solution of (1), we can prove that the resolvent kernel \( \Gamma(s, t; \lambda) \) is unique. In fact, let equation (1) have, with \( \lambda = \lambda_0 \), two resolvent kernels \( \Gamma_1(s, t; \lambda_0) \) and \( \Gamma_2(s, t; \lambda_0) \). In view of the uniqueness of the solution of (1), an arbitrary function \( f(s) \) satisfies the identity
\[
f(s) + \lambda_0 \int \Gamma_1(s, t; \lambda_0) f(t) \, dt = f(s) + \lambda_0 \int \Gamma_2(s, t; \lambda_0) f(t) \, dt \quad (27)
\]
Setting \( \Psi(s, t; \lambda_0) = \Gamma_1(s, t; \lambda_0) - \Gamma_2(s, t; \lambda_0) \), we have from (27)

\[
\int \Psi(s, t; \lambda_0) f(t) \, dt = 0
\]

For an arbitrary function \( f(t) \). Let us take \( f(t) = \Psi^*(s, t; \lambda_0) \) with fixed \( s \) this implies that

\[
\int |\Psi(s, t; \lambda_0)|^2 \, dt = 0
\]

Which means that \( \Psi(s, t; \lambda_0) = 0 \), proving the uniqueness of the resolvent kernel. The above analysis can be summed up in the following basic theorem

**Theorem.** To each \( L_2 \)-kernel \( k(s, t) \), there corresponds a unique resolvent kernel \( \Gamma(s, t; \lambda) \) which is an analytic function of \( \lambda \), regular at least inside the circle \( |\lambda| < B^{-1} \), and represented by the power series (26). Furthermore, if \( f(s) \) is also an \( L_2 \)-function of the fredholm equation (1) valid in the circle \( |\lambda| < B^{-1} \) is given by the formula (25). The method of successive approximations has many drawbacks. In addition to begin a cumbersome process, the Neumann series, in general, cannot be summed in closed form. Furthermore, a solution of the integral equation (1) may exit even if \( |\lambda|B > 1 \), as evidenced in the previous chapter. In fact we saw that the resolvent kernel is a quotient of two polynomials of nth degree in \( \lambda \), and therefore the only possible singular points of \( \Gamma(s, t; \lambda) \) are the roots of the denominator \( D(\lambda) = 0 \) but for \( |\lambda|B > 1 \) the Neumann series does not converge and as such does not provide the desired solution. We shall have more to say about these ideas in the next chapter.

### 2.2 Examples

#### 2.2.1 Example 1

Solve the integral equation

\[
g_1(s) = f(s) + \lambda \int e^{s-t} g(t) \, dt \tag{1}
\]

Following the method of the previous section, we have

\[
k_1(s, t) = e^{s-t}
\]

\[
k_2(s, t) = \int_0^1 e^{s-x} e^{x-t} \, dx = e^{s-t}
\]
Proceeding in the way, we find that all the iterated kernels coincide with $k(s, t)$

Using (3.1.26), we obtain the resolvent kernel as

$$
\Gamma(s; \lambda) = k(s, t)(1 + \lambda + \lambda^2 + \cdots) = \frac{e^{s-t}}{1 - \lambda}
$$

(2)

Although the series $\left(1 + \lambda + \lambda^2 + \cdots\right)$ converges only for $|\lambda| < 1$ the resolvent kernel is, in fact an analytic function of $\lambda$ regular in the whole plane except at the point $\lambda = 1$ which is a simple pole of the kernel $\Gamma$ the solution $g(s)$ then follows from (3.1.25):

$$
g(s) = f(s) - \left[\lambda/(\lambda - 1)\right] \int_0^1 e^{s-t} \ g(t) \ dt
$$

(3)

2.2.2 Example 2 solve the fredholm integral equation

$$
g(s) = 1 + \lambda \int_0^1 (1 - 3st) \ g(t) \ dt
$$

(4)

And evaluate the resolvent kernel. Starting with $g_0(s) = 1$, we have

$$
g_1(s) = 1 + \lambda \int_0^1 (1 - 3st) \ dt = 1 + \lambda \left(1 - \frac{3}{2}s\right)
$$

$$
g_2(s) = 1 + \lambda \int_0^1 (1 - 3st) \left[1 + \lambda \left(1 - \frac{3}{2}s\right)\right] \ dt = 1 + \lambda \left(1 - \frac{3}{2}s\right) + \frac{1}{4} \lambda^2
$$

$$
g(s) = 1 + \lambda \left(1 - \frac{3}{2}s\right) + \frac{1}{4} \lambda^2 + \frac{1}{4} \lambda^3 \left(1 - \frac{3}{2}s\right) + \frac{1}{16} \lambda^4 + \frac{1}{16} \lambda^5 \left(1 - \frac{3}{2}s\right) + \cdots
$$

Or

$$
g(s) = \left(1 + \frac{1}{4} \lambda^2 + \frac{1}{16} \lambda^4 + \cdots\right) \left[1 + \lambda \left(1 - \frac{3}{2}s\right)\right]
$$

(5)

The geometric series in (4) is convergent provided $|\lambda| < 2$ then

$$
g(s) = \left[4 + 2\lambda(2 - 3s)\right]/(4 - \lambda^2)
$$

(6)

And precisely the same remarks apply to the region of the validity of this solution as given in Example 1.

To evaluate the resolvent kernel, we find the iterated kernels $k_1(s, t) = 1 - 3st$
\[ k_1(s, t) = 1 - 3st \]
\[ k_2(s, t) = \int_0^1 (1 - 3sx)(1 - 3xt) \, dx = 1 - \frac{3}{2}(s + t) + 3st \]
\[ k_3(s, t) = \int_0^1 (1 - 3sx)[1 - \frac{3}{2}(x + t) + 3xt] \, dx \]
\[ = \frac{1}{4}(1 - 3st) = \frac{1}{4}k_1(s, t) \]

Similarly
\[ k_4(s, t) = \frac{1}{4}k_1(s, t) \]

And
\[ k_n(s, t) = \frac{1}{4}k_{n-2}(s, t) \]

Hence
\[ \Gamma(s, t; \lambda) = k_1 + \lambda k_2 + \lambda^2 k_3 + \cdots \]
\[ = \left(1 + \frac{1}{4}\lambda^2 + \frac{1}{16}\lambda^4 + \cdots\right)k_1 + \lambda \left(1 + \frac{1}{4}\lambda^2 + \frac{1}{16}\lambda^4 + \cdots\right)k_2 \]
\[ = \left[\frac{(1 + \lambda) - \frac{3}{2}\lambda(s + t) - 3(1 - \lambda)st}{1 - \frac{1}{4\lambda^2}}\right], \quad |\lambda| < 2 \quad (7) \]
2.2.3 Example 3

solve the integral equation

\[
g(s) = 1 + \lambda \int_0^\pi [(\sin(s + t)] \ g(t) \ dt \quad (8)
\]

Let us first evaluate the iterated kernels in this example:

\[
k_1(s, t) = k(s, t) = \sin(s + t)
\]

\[
k_2(s, t) = \int_0^\pi [(\sin(s + x)] \ \sin(x + t) \ dx
\]

\[
= \frac{1}{2} \pi \sin s \sin t + \cos s \cos t = \frac{1}{2} \pi \cos(s - t)
\]

\[
k_3(s, t) = \frac{1}{2} \pi \int_0^1 [(\sin(s + x)] \ \cos(x - t) \ dx
\]

\[
= \left(\frac{1}{2} \pi\right)^2 [\sin s \cos t + \cos s \sin t] = \left(\frac{1}{2} \pi\right)^2 \sin(s + t)
\]

Proceeding in this manner, we obtain

\[
k_4(s, t) = \left(\frac{1}{2} \pi\right)^3 \cos(s - t),
\]

\[
k_5(s, t) = \left(\frac{1}{2} \pi\right)^4 \sin(s + t),
\]

\[
k_6(s, t) = \left(\frac{1}{2} \pi\right)^5 \cos(s - t), \quad \text{ etc.}
\]

Substituting these values in the formula (3.1.13) and integrating their results the solution

\[
g(s) = 2\lambda \ (\cos s) \left[ 1 + \left(\frac{1}{2} \pi\right)^2 \lambda^2 + \left(\frac{1}{2} \pi\right)^4 \lambda^4 + \cdots \right]
\]

\[
+ \lambda^2 \pi(\sin s) \left[ 1 + \left(\frac{1}{2} \pi\right)^2 \lambda^2 + \left(\frac{1}{2} \pi\right)^4 \lambda^4 + \cdots \right] \quad (9)
\]

Or
\[ g(s) = 1 + \frac{2\lambda \cos s + \lambda^2 \pi \sin s}{1 - \frac{1}{4} \lambda^2 \pi^2} \quad (10) \]

Since

\[ B^2 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sin^2 (s + t) \, ds \, dt = \frac{1}{2} \pi^2 \]

The interval of convergence of the series (9) lies between \(-\sqrt{2}\pi\) and \(\sqrt{2}\pi\)

2.2.4 Examples 4 prove that the mth iterated kernel \( k_m(s, t) \) satisfies the following relation:

\[ k_m(s, t) = \int k_r(s, x) \, k_{m-r}(x, t) \, dx \quad (11) \]

Where \( r \) is any positive integer less than \( m \) by successive applications of (3.1.11)

\[ k_m(s, t) = \int \cdots \int k(s, x_1) \, k(x_1, x_2) \cdots k(x_{m-1}, t) \, dx_{m-1} \cdots dx \quad (12) \]

Thus \( k_m(s, t) \) is an \((m - 1)\) fold integral. Similarly

\[ k_r(s, t) \text{ and } k_{m-r}(s, t) \text{ are } (r - 1)\text{and } (m - r - 1)\text{fold} \]

Fold integrals. This means that

\[ \int k_r(s, x) \, k_{m-r}(x, t) \, dx \]

Is a \((m - 1)\)fold integral, and the result follows one can take the \( g_0(s) \) approximation different from \( f(s) \) as we demonstrate by the following example.
2.2.5 Example 5  solve the inhomogeneous fredholm integral equation of the second kind

\[
g(s) = 2s + \lambda \int_0^1 (s + t) g(t) \, dt \tag{13}
\]

By the method of successive approximations to the third order for this equation, we take \(g_0(s) = 1\). Then

\[
g(s) = 2s + \lambda \int_0^1 (s + t) \, dt = 2s + \lambda (s + \frac{1}{2})
\]

\[
g_2(s) = 2s + \lambda \int_0^1 (s + t) \left\{ 2t + \lambda \left[ t + \frac{1}{2} \right] \right\} \, dt = 2s + \lambda \left[ s + \left( \frac{2}{3} \right) \right] + \lambda^2 \left[ s + \left( \frac{7}{12} \right) \right]
\]

\[
g_3(s) = 2s + \lambda \int_0^1 (s + t) \left\{ 2t + \lambda \left[ t + \frac{2}{3} \right] + \lambda^2 \left[ t + \frac{7}{12} \right] \right\} \, dt
\]

\[
= 2s + \lambda \left[ s + \left( \frac{2}{3} \right) \right] + \lambda^2 \left[ s + \left( \frac{7}{6} \right) + \left( \frac{2}{3} \right) \right] + \lambda^3 \left[ \left( \frac{13}{12} \right) s + \left( \frac{5}{8} \right) \right] \tag{14}
\]

From example 2 of section 2.2, we find the exact solution to be

\[
g(s) = \frac{12(2 - \lambda)s + 8\lambda}{12 - 12\lambda - \lambda^2} \tag{15}
\]

And the comparison of (14) and (15) is left to the reader.
2.3 Volterra integral equation

The same iterative scheme is applicable to the volterra integral equation of the second kind. In fact, the formulas corresponding (3.1.13) and (3.1.25) are, respectively,

\[ g(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int_{a}^{s} k_m(s, t) \ f(t) \ dt \] \hspace{1cm} (1)

\[ g(s) = f(s) + \lambda \int_{a}^{s} \Gamma(s, t; \lambda) \ f(t) \ dt \] \hspace{1cm} (2)

Where the iterated kernel \( k_m(s, t) \) satisfies the recurrence formula

\[ k_m(s, t) = \int_{t}^{s} k(s, x) \ k_{m-1}(x, t) \ dx \] \hspace{1cm} (3)

With \( k_1(s, t) = k(s, t) \) as before the resolvent kernel \( \Gamma(s, t; \lambda) \) is given by the same formula as (3.1.26), and it is an entire function of \( \lambda \) for any given \((s, t)\)

2.4 EXAMPLES

2.4.1 Example 1  Find the Neumann series for the solution of the integral equation

\[ g(s) = (1 + s) + \lambda \int_{0}^{s} (s - t) \ g(t) \ dt \] \hspace{1cm} (1)

From the formula (3.3.3) we have

\[ k_1(s, t) = (s - t) \]

\[ k_2(s, t) = \int_{t}^{s} (s - x) \ (x - t) \ dx = \frac{(s - t)^3}{3!} \]

\[ k_3(s, t) = \int_{t}^{s} \frac{(s - x)(x - t)^3}{3!} \ dx = \frac{(s - t)^5}{5!} \]

And so on. Thus

\[ g(s) = 1 + s + \lambda \left( \frac{s^2}{2!} + \frac{s^3}{3!} \right) + \lambda^2 \left( \frac{s^4}{4!} + \frac{s^5}{5!} \right) + ... \] \hspace{1cm} (2) \hspace{1cm} \text{for } \lambda = 1, \hspace{1cm} g(s) = e^s
2.4.2 Example 2  solve the integral equation

\[ g(s) = f(s) + \lambda \int_0^s e^{s-t} g(t) \, dt \]  \hspace{1cm} (3)

And evaluate the resolvent kernel.

For this case

\[ k_1(s, t) = e^{s-t} \]

\[ k_2(s, t) = \int_t^s e^{s-x} e^{x-t} \, dx = (s-t)e^{s-t} \]

\[ k_2(s, t) = \int_t^s e^{s-x} e^{x-t} \, dx = (s-t)e^{s-t} \]

\[ k_3(s, t) = \int_t^s (x-t)e^{s-x} e^{x-t} \, dx = \frac{(s-t)^2}{2!} e^{s-t} \]

\[ k_m(s, t) = \frac{(s-t)^{m-1}}{(m-1)!} e^{s-t} \]

The resolvent kernel is

\[ \Gamma(s, t; \lambda) = \begin{cases} 
  e^{s-t} \sum_{m=1}^{\infty} \frac{\lambda^{m-1}(s-t)^{m-1}}{(m-1)!}, & t \leq s \\
  0, & t > s 
\end{cases} \]  \hspace{1cm} (4)

Hence, the solution is

\[ g(s) = f(s) + \lambda \int_0^s e^{(\lambda+1)(s-t)} g(t) \, dt \]  \hspace{1cm} (5)
2.4.3 Example 3  

Solve the Volterra equation

\[ g(s) = 1 + \int_0^s g(t) \, dt \quad (6) \]

For this example

\[ k_1(s, t) = k(s, t) = st \]
\[ k_2(s, t) = \int_t^s s^2 x^2 \, dx = \frac{s^4 t - s^4}{3} \]
\[ k_3(s, t) = \int_t^s (sx)(x^4 t - xt^4) \, dx = \frac{s^7 t - 2s^4 t^4 + st^7}{18} \]
\[ k_4(s, t) = \int_t^s (sx)(x^7 t - 2x^4 t^4 + xt^7) \, dx = \frac{s^{10} t - 3s^7 t^4 + 3s^4 t^7 - st^{10}}{162} \]

And so on thus.

\[ g(s) = 1 + \frac{s^3}{3} + \frac{s^6}{2.5} + \frac{s^9}{2.5.8} + \frac{s^{12}}{2.5.8.11} + \cdots \quad (7) \]
2.5 SOME RESULTS ABOUT THE RESOLVENT KERNEL

The series for the resolvent kernel $\Gamma(s, t; \lambda)\n\Gamma(s, t; \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} \ k_m(s, t) \quad (1)\n
Can be proved to be absolutely and uniformly convergent for all values of $s$ and $t$ in the circle $|\lambda| < \frac{1}{B}$ in addition to the assumptions of section 3.1 we need the additional inequality
\[ \int |k(s, t)|^2 \ ds < E^2 \, , \quad E = \text{const} \quad (2)\n
Recall that this is one of the conditions for the kernel $K$ to be an $L_2$-kernel

Applying the Schwarz inequality to the recurrence formula
\[ k_m(s, t) = \int k_{m-1}(s, x) k(x, t) \ dx \quad (3)\n
Yields
\[ |k_m(s, t)|^2 \leq \left( \int |k_{m-1}(s, x)|^2 \ dx \right) \int |k(x, t)|^2 \ dx \n\]
Which, with the help of [1], becomes
\[ |k_m(s, t)| \leq C_E B^{m-1} \quad (4)\n
Thus, the series (1) is dominated by the geometric series with the general term $C_E (\lambda^{m-1} B^{m-1})$, and that completes the proof.

Next, we prove that the resolvent kernel satisfies the integral equation this follows by replacing $k_m(s, t)$ in the series (1) by the integral relation (3) then
\[ \Gamma(s, t; \lambda) = k_1(s, t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_a^s k_{m-1}(s, x) k(x, t) \ dx \]
\[ = k(s, t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^s k_m(s, x) k(x, t) \ dx \]
\[ = k(s, t) + \lambda \int \left[ \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s, x) \right] k(x, t) \, dx \]

And the integral equation (5) follows immediately. The change of order of integration and summation is legitimate in view of the uniform convergence of the series involved.

Another interesting result for the resolvent kernel is that it satisfies the integrodifferential equation

\[ \frac{\partial \Gamma(s, t; \lambda)}{\partial \lambda} = \int \Gamma(s, x; \lambda) \Gamma(x, t; \lambda) dx \quad (6) \]

In fact

\[ \int \Gamma(s, x; \lambda) \Gamma(x, t; \lambda) dx = \int \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s, x) \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x, t) \, dx \]

On account of the absolute and uniform convergence of the series (1) we can multiply the series under the integral sign and integrate it term by term. Therefore,

\[ \int \Gamma(s, x; \lambda) \Gamma(x, t; \lambda) dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda^{m+n-2} k_{m+n}(s, t) \quad (7) \]

Now, set \( m + n = p \) and change the order of summation; there results

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda^{m+n-2} k_{m+n}(s, t) = \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} \lambda^{p-2} k_p(s, t) \]

\[ \sum_{p=2}^{\infty} (p-1) \lambda^{p-2} k_p(s, t) = \frac{\partial \Gamma(s, t; \lambda)}{\partial \lambda} \quad (8) \]

Combining (7) and (8), we have the result (6).
3.1 INITIAL VALUE PROBLEMS

There is a fundamental relationship between Volterra integral equations and ordinary differential equations with prescribed initial values. We begin our discussion by studying the simple initial value problem

\[ y'' + A(s)y' + B(s)y = F(s) \quad (1) \]
\[ y(a) = q_0, \; y'(a) = q_1 \quad (2) \]

Where a prime implies differentiation with respect to \( s \), and the functions \( A, B \) and \( F \) are defined and continuous in the closed interval \( a \leq s \leq b \). The result of integrating the differential equation (1) from \( a \) to \( s \) and using the initial values (2) is:

\[ y'(s) - q_1 = -A(s)y(s) - \int_{0}^{s} [B(s_1) - A'(s_1)]y(s_1)ds_1 + \int_{a}^{s} F(s_1)ds_1 + A(a)q_0 \]

Similarly, a second integration yields

\[ y(s) - q_0 = -\int_{a}^{s} A(s_1)y(s_1)ds_1 - \int_{a}^{s} \int_{a}^{s_2} [B(s_1) - A'(s_1)]y(s_1)ds_1ds_2 + \int_{a}^{s} \int_{a}^{s_2} F(s_1)ds_1ds_2 + [A(a)q_0 + q_1](s - a) \quad (3) \]

With the help of the identity (see Appendix, section A.1)

\[ \int_{a}^{s} \int_{a}^{s_2} F(s_1)ds_1ds_2 = \int_{a}^{s} (s - t)(t)dt \quad (4) \]

The two double integrals in (3) can be converted to single integrals hence; the relation (3) takes the form
\[ y(s) = q_0 + [A(a)q_0 + q_1](s - a) + \int_a^s (s - t)F(t)dt - \int_a^s \{A(t) + (s - t)[B(t) - A'(t)]\}y(t)dt \quad (5) \]

Now, set
\[ K(s,t) = -\{A(t) + (s - t)[B(t) - A'(t)]\} \quad (6) \]

And
\[ f(s) = \int_a^s (s - t)F(t)dt + [A(a)q_0 + q_1](s - t) + q_0 \quad (7) \]

From relations (5)-(7), we have the Volterra integral equation of the second kind:
\[ y(s) = f(s) + \int_a^s K(s,t)y(t)dt \quad (8) \]

Conversely, any solution \( g(s) \) of the integral equation (8) is, as can be verified by two differentiations, a solution of the initial value problem (1)-(2).

Note that the crucial step is the use of the identity (4). Since we have proved the corresponding identity for an arbitrary integer \( n \) in the Appendix, section A.1, it follows that the above process of converting an initial value problem to a Volterra integral equation is applicable to a linear ordinary differential equation of order \( n \) when there are \( n \) prescribed initial conditions. An alternative approach is somewhat simpler for proving the above-mentioned equivalence for a general differential equation. Indeed, let us consider the linear differential equation of order \( n \):
\[ \frac{d^n y}{ds^n} + A_1(s)\frac{d^{n-1} y}{ds^{n-1}} + \cdots + A_{n-1}(s)\frac{dy}{ds} + A_n(s)y = F(s) \quad (9) \]

With the initial conditions
\[ y(a) = q_0, \quad y'(a) = q_1, \quad \ldots, \quad y^{(n-1)}(a) = q_{n-1} \quad (10) \]

Where the functions \( A_1, A_2, \ldots, A_n \) and \( F \) are defined and continuous in \( a \leq s \leq b \).

The reduction of the initial value problem (9)-(10) to the Volterra integral equation is accomplished by introducing an unknown function \( g(s) \):
\[ d^n y|ds^n = g(s) \quad (11) \]

From (10) and (11), it follows that
\[
\frac{d^{n-1}(y)}{ds^{n-1}} = \int_a^s g(t) dt + q_{n-1} \quad (12)
\]

\[
\frac{d^{n-2}y}{ds^{n-2}} = \int_a^s (s-t)g(t) dt + (s-a)q_{n-1} + q_{n-2} \quad \text{(continued) ...}
\]

\[
\frac{dy}{ds} = \int_a^s \frac{(s-t)^{n-2}}{(n-2)!} g(t) dt + \frac{(s-t)^{n-2}}{(n-1)!} q_{n-1} + \frac{(s-t)^{n-3}}{(n-3)!} q_{n-2} + \cdots + (s-a)q_2 + q_1 \quad (12)
\]

\[
Y = \int_a^s \frac{(s-t)^{n-1}}{(n-1)!} g(t) dt + \frac{(s-a)^{n-1}}{(n-1)!} q_{n-1} + \frac{(s-a)^{n-2}}{(n-2)!} q_{n-2} + \cdots + (s-a)q_2 + q_1
\]

Now, if we multiply relations (11) and (12) by \(1, A_1(s), A_2(s), \text{etc. and add, we find that the initial value problem defined by (9)-(10) is reduced to the Volterra integral equation of the second kind}

\[
g(s) = f(s) + \int_a^s K(s,t) g(t) dt \quad (13)
\]

Where

\[
K(s,t) = - \sum_{k=1}^n A_k(s) \frac{(s-t)^{k-1}}{(k-1)!} \quad (14)
\]

\[
f(s) = F(s) - q_{n-1}A_1(s)[(s-a)q_{n-1} + q_{n-2}]A_2(s) - \cdots
\]

\[
- \left\{ \frac{(s-a)^{n-1}}{(n-1)!} q_{n-1} + \cdots + (s-a)q_1 + q_0 \right\} \times A_n(s) \quad (15)
\]

Conversely, if we solve the integral equation (13) and substitute the value obtained for \(g(s)\) in the last equation of the system (12), we derive the (unique) solution of the initial value problem (9)-(10), view [1]
3.2 BOUNDARY VALUE PROBLEMS

Just as initial value problems in ordinary differential equations lead to Volterra-type integral equations, boundary value problems in ordinary differential equations lead to Fredholm-type integral equations let us illustrate this equivalence by the problem

\[ y'' + A(s)y' + B(s)y = F(s) \quad (1) \]
\[ y(a) = y_0, \quad y'(a) = y_1 \quad (2) \]

When we integrate equation (1) from \( a \) to \( s \) and use the boundary condition \( y(a) = y_0 \), we get

\[ y'(s) = C + \int_a^s F(s)ds - A(s)y(s) + A(a)y_0 + \int_a^s [A'(s) - B(s)]y(s)ds, \]

Where \( C \) is a constant of integration. a second integration similarly yields

\[ y(s) - y_0 = [C + A(a)y_0](s - a) + \int_a^s \int_a^{s2} F(s_1)ds_1 ds_2 - \int_a^s A(s_1)y(s_1)ds_1 + \int_a^s \int_a^{s2} [A'(s_1) - B(s_1)]y(s_1)ds_1 ds_2 \quad (3) \]

Using the identity (5.1.4), the relation (3) becomes

\[ y(s) - y_0 = [C + A(a)y_0](s - a) + \int_a^s (s - t)F(t)dt - \int_a^s \{A(t) - (s - t)[A'(t) - B(t)]\} - y(t)dt \quad (4) \]

The constant \( C \) can be evaluated by setting \( s = b \) in (4) and using the second boundary condition \( y(b) = y_1 \):

\[ y_1 - y_0 = [C + A(a)y_0](s - a) + \int_a^s (s - t)F(t)dt - \int_a^s \{A(t) - (s - t)[A'(t) - B(t)]\} - y(t)dt \]

Or

\[ C + A(a)y_0 = \left[ \frac{1}{b - a} \right] \{y_1 - y_0\} - \int (b - t)F(t)dt + \int \{A(t) - (b - t)[A'(t) - B(t)]\}y(t)dt \quad (5) \]
From (4) and we have the relation

\[
y(s) = y_0 + \int_a^s (s - t)F(t)dt + \left[\frac{s - a}{b - a}\right] \times [(y_1 - y_0) - \int (b - t)F(t)dt] - \int \{A(t) - (s - t)[A'(t) - B(t)]\} y(t)dt + \int \left[\frac{s - a}{b - a}\right] \{A(t) - (b - t)[A'(t) - B(t)]\} y(t)dt
\]  

(6)

Equation (6) can be written as the Fredholm integral equation

\[
y(s) = f(s) + \int K(s, t)y(t)dt \quad (7)
\]

Provided we set

\[
f(s) = y_0 + \int_a^s (s - t)F(t)dt + \left[\frac{s - a}{b - a}\right] [(y_1 - y_0) - \int (b - t)F(t)dt]
\]  

(8)

And

\[
K(s, t) = \begin{cases} 
\left[\frac{s - a}{b - a}\right] \{A(t) - (b - t)[A'(t) - B(t)]\}, & s < t. \\
A(t) \left[\frac{s - a}{b - a} - 1\right] - [A'(t) - B(t)] \times \left[\frac{(t - a)(b - s)}{b - a}\right], & s > t.
\end{cases}
\]  

(9)

For the special case when A and B are constants, \(a=0, b=1\), and \(y(0) = y(1) = 0\), the above kernel simplifies to

\[
k(s, t) = \begin{cases} 
Bs(1 - t) + As, & s < t. \\
Bt(1 - s) + As - A, & s > t.
\end{cases}
\]  

(10)

Note that the kernel (10) is asymmetric and discontinuous at \(s=t\), unless \(A=0\). We shall elaborate on this point

**3.3 EXAMPLES**, view [3]

**3.3.1 Example 1** Reduce the initial value problem

\[
y^n(s) + \lambda y(s) = F(s), \quad (1)
\]

\[
y(0) = 1, \quad y^{(0)} = 0, \quad (2)
\]

To a Volterra integral equation
Comparing (1) and (2) with the notation of Section 5.1, we have
A(s) = 0, B(s) = \lambda. Therefore, the relations (5.1.6)-(5.1.8) become

\[ K(s, t) = \lambda(t - s), \]
\[ f(s) = 1 + \int_0^s (s - t)F(t)dt \quad (3) \]

And
\[ y(s) = 1 + \int_0^s (s - t)F(t)dt + \lambda \int_0^s (t - s)y(t)dt \]

3.3.2 Example 2  Reduce the boundary value problem

\[ y''(s) + \lambda P(s)y = Q(s) \quad (4) \]
\[ y(a) = 0, \quad y(b) = 0 \quad (5) \]

To a Fredholm integral equation.

Comparing (4) and (5) with the notation of Section 5.2 in [1] we have A=0, B=\lambda P(s), F(s)=Q(s), y_0 = 0, y_1 = 0 substitution value in the relations (5.2.8) and (5.2.9) yields

\[ f(s) = \int_a^s (s - t)Q(t)dt - \left[ \int_a^{s-a} \right] \int (b - t)Q(t)dt \quad (6) \]

And

\[ K(s, t) = \begin{cases} \lambda P(t) \left[ \frac{(s - a)(b - t)}{b - a} \right], & s < t \\ \lambda P(t) \left[ \frac{(t - a)(b - s)}{b - a} \right], & s > t \end{cases} \quad (7) \]

Which, when put in (5.2.7), gives the required integral equation. Note that the kernel is continuous at s = t.

As a special case of the above example, let us take the boundary value problem

\[ y'' + \lambda y = 0, \quad (8) \]
\[ y(0) = 0, \quad y(\ell) = 0 \quad (9) \]

Then, the relations (6) and (7) take the simple forms: f(s) = 0, and

\[ K(s, t) = \begin{cases} \frac{\lambda s}{\ell} (\ell - t), & s < t. \\ \frac{\lambda t}{\ell} (\ell - s), & s > t. \end{cases} \quad (10) \]
Note that, although the kernels (7) and (10) are continuous at \( s = t \), their derivatives are not continuous. For example, the derivative of the kernel (10) is

\[
\frac{\partial K(s, t)}{\partial s} = \begin{cases} 
    \lambda \left[1 - \left(\frac{t}{\ell}\right)\right], & s < t, \\
    -\frac{\lambda t}{\ell}, & s > t.
\end{cases}
\]

The value of the jump of this derivative at \( s = t \) is

\[
\left[ \frac{dK(s, t)}{ds} \right]_{s=t+0} - \left[ \frac{dK(s, t)}{ds} \right]_{s=t-0} = -\lambda
\]

Similarly, the value of the jump of the derivative of the

Kernel (7) at \( s = t \) is

\[
\left[ \frac{dK(s, t)}{ds} \right]_{s=t+0} - \left[ \frac{dK(s, t)}{ds} \right]_{s=t-0} = -\lambda P(t)
\]

### 3.3.3 Example 3

Transverse oscillations of a homogeneous elastic bar.
Consider a homogeneous elastic bar with linear mass density \( d \). Its axis coincides with the segment \((0, \ell)\) of the \( s \) axis when the bar is in its state of rest. It is clamped at the end \( s = 0 \), free at the end \( s = \ell \), and is forced to perform simple harmonic oscillations with period \( \frac{2\pi}{\omega} \). The problem to find the deflection \( y(s) \) that is parallel to the \( y \) axis and satisfies the system of equations

\[
\frac{d^4y}{ds^4} - K^4y = 0, \quad K^4 = \frac{\omega^2 d}{EI}, \quad (11)
\]

\[
y(0) = 0, \quad y'(0) = 0 \quad (12)
\]

\[
y''(t) = y'''(t) = 0 \quad (13)
\]

Where \( EI \) the bending rigidity of the bar. The differential equation (11) with the initial conditions (12) can be reduced to the solution of a Volterra integral equation if we stipulate that

\[
y''(0) = C_2, \quad y'''(0) = C_3, \quad (14)
\]

And subsequently determine the constants \( C_2 \) and \( C_3 \) with the help of (13). Indeed, when we compare the initial value problem embodied in (11), (12) and (14) with the system (5.1.9)-(5.1.15) in [1], we obtain the required integral equation

\[
g(s) = K^4 \left( \frac{s^2}{2!} C_2 + \frac{s^3}{3!} C_3 \right) + K^4 \int_a^s \frac{(s-t)^3}{3!} g(t)dt, \quad (15)
\]
Where

\[ g(s) = \frac{d^4y}{ds^4} \quad (16) \]

The solution \( y(s) \) of the differential equation (11) is

\[
y(s) = \int_0^s (s - t)^3 \frac{3!}{3!} g(t) t + \frac{s^3}{3!} C_3 + \frac{s^2}{2!} C_2 \quad (17)
\]

We leave it to the reader to apply the conditions (13), determine the constants \( C_2 \) and \( C_3 \), and thereby complete the transformation of the system (11)-(13) into an integral equation. The kernel of the integral equation (15) is that of convolution type and this equation can be readily solved by Laplace transform methods as explained in chapter 9 (see Exercise 11 at the end of that chapter).

### 3.4 DIRAC DELTA FUNCTION

In physical problems, one often encounters idealized concepts such as a unit force acting at a point \( s = s_0 \) only, or an impulsive force acting at time \( t = t_0 \) only. These forces are described by the Dirac delta function \( \delta(s - s_0) \) or \( (t = t_0) \) such that

\[
\delta(x - x_0) = \begin{cases} 
0 & , \quad x \neq x_0 \\
\infty & , \quad x = x_0 
\end{cases} \quad (1)
\]

Where \( x \) stands for \( s \) in the case of a unit force and \( t \) in the case of an impulsive force. Also,

\[
\int_{-\infty}^{\infty} \delta(x - x_0) \varnothing(x) dx = \varnothing(x_0) \quad (2)
\]

For every continuous function \( \varnothing(x) \), the Dirac delta function has been successfully used in the description of concentrated forces in solid and fluid mechanics, point masses in the theory of gravitational potential, point charges in electrostatics, impulsive forces in acoustics, and various similar phenomena in other branches of physics and mechanics.

In spite of the fact that scientists have used this function with success; the language of classical mathematics is inadequate to justify such a function. It is usually visualized as a limit of piecewise continuous functions \( f(x) \) such as

\[
f(x) = \begin{cases} 
0 & , \quad 0 \leq x < x_0 - \frac{1}{2} \varepsilon, \\
P & , \quad |x - x_0| \leq \frac{1}{2} \varepsilon, \\
0 & , \quad x_0 + \frac{1}{2} \varepsilon < x < l ,
\end{cases} \quad (4)
\]
Or as a limit of a sequence of suitable functions such as

\[ f_k(x) = \begin{cases} 
K, & 0 < |x| < 1/k \\
0, & \text{for all other } x
\end{cases} \]  

(5)

Where \( k = 1, 2, 3, \ldots \), and

\[ f_k(x) = \frac{1}{\pi} \frac{\sin k x}{x} \]  

(6)

Our aim in this and the next chapter is to determine integral representation formulas for the solutions of linear ordinary and partial differential equations in such a manner so as to include the boundary conditions explicitly. To accomplish this task, we have to solve differential equations whose inhomogeneous term is a concentrated source. This is best done by introducing the theory of distributions, but that is not on our agenda. We shall, therefore, content ourselves with the above-mentioned properties of the delta function. Furthermore, we shall need the Heaviside function \( H(x) \):

\[ H(x) = \begin{cases} 
0, & x < 0 \\
1, & x > 0
\end{cases} \]  

(7)

And the relation

\[ \frac{dH(x)}{dx} = \delta(x) \]  

(8)
3.5 GREEN'S FUNCTION APPROACH

We shall consider the initial and boundary value problems of sections 5.1 and 5.2 in [1] a different context. Let $L$ be the differential operator

$$Lu(s) = \left[ A(s) \frac{d^2}{ds^2} + B(s) \frac{d}{ds} + C(s) \right] u(s), \quad a < s < b \quad (1)$$

Where $A(s)$ is continuously differentiable, positive function. Its adjoint operator $M$ is defined as

$$Mv(s) = \frac{d^2}{ds^2} [A(s)v(s)] - \frac{d}{ds} [B(s)v(s)] + C(s)v(s), \quad a < s < b \quad (2)$$

It follows by integration by parts that

$$\int (vLu - uMv)ds = [A(vu' - uv') + uv(B - A')]_a^b \quad (3)$$

This is known as Green's formula for the operator $L$. It is traditionally proved in the theory of ordinary differential equations that the relation

$$A(s)y'' + B(s)y' + C(s)y = \Phi(s) \quad (4)$$

Can be converted to the form

$$\frac{d}{ds} \left( p(s) \frac{dy}{ds} \right) + q(s)y = F(s) \quad (5)$$

Which is clearly self-adjoint. The function $p(s)$ id again continuously differentiable and positive and $q(s)$ and $F(s)$ are continuous in a given interval $(a, b)$. Green's formula (3) for this operator take the simple form

$$\int (vLu - uMv)ds = [p(s)(vu' - uv')]_a^b \quad (6)$$

The homogeneous second-order equation

$$\frac{d}{ds} \left( p \frac{dy}{ds} \right) + qy = 0 \quad (7)$$

Has exactly two linearly independent solutions $u(s)$ and $v(s)$ which are twice continuously differentiable in the interval $a < s < b$. Any other solution of this equation is a linear combination of $u(s) and v(s)$, i.e., $y(s) = c_1 u(s) + c_2 v(s)$, where $c_1$ and $c_2$ are constants.
3.5.1 Initial value problems

Let us first consider the initial value problem

\[
\frac{d}{ds} \left( p \frac{dy}{ds} \right) + qy = F(s),
\]

(8)

\[y(a) = 0, \quad y'(a) = 0 \quad \text{(9)}\]

To formulate this problem into an integral equation, we consider the function

\[
w(s) = u(s) \int_a^s v(t)F(t)dt - v(s) \int_a^s u(t)F(t)dt,
\]

(10)

Where \(u\) and \(v\) are solution of the homogeneous equation (7) as mentioned above. The relation (10), when differentiated, gives

\[
w'(s) = u'(s) \int_a^s v(t)F(t)dt
\]

\[\quad - v'(s) \int_a^s u(t)F(t)dt + u(s)v(s)F(s) - u(s)v(s)F(s)
\]

\[= u'(s) \int_a^s v(t)F(t)dt - v'(s) \int_a^s u(t)F(t)dt
\]

Hence, \(w(a) = w'(a) = 0\), and

\[
\frac{d}{ds} \left[ p(s) \frac{dw}{ds} \right] = \frac{d}{ds} \left[ p(s) \frac{du}{ds} \right] \int_a^s v(t)F(t)dt - \frac{d}{ds} \left[ p(s) \frac{dv}{ds} \right] \int_a^s u(t)F(t)dt
\]

\[\times \int_a^s u(t)F(t)dt + p(s)[u'(s)v(s) - v'(s)u(s)]F(s)
\]

\[= -q(s)w(s) + p(s)[u'(s)v(s) - v'(s)u(s)]F(s), \quad \text{(11)}
\]

Where we have used the fact that \(u(s)\) and \(v(s)\) satisfy the equation (7).

In addition [dropping the argument \((s)\) for \(p, u, v\),

\[
\frac{d}{ds} \left\{ p(u'v - v'u) \right\} = \frac{d}{ds} (pu')v - \frac{d}{ds} (pv')u + pu'v - pv'u' = 0,
\]

Also because \(u\) and \(v\) satisfy (7). This means that

\[
p(s)[u'(s)v(s) - v'(s)u(s)] = A \quad \text{(12)}
\]
Where \( A \) is a constant. the negative of the expression in the brackets in the above relation is called the Wronskian \( w(u,v; s) \) of the solution \( u \) and \( v \):

\[
w(u,v; s) = u(s)v'(s) - v(s)u'(s) \quad (13)
\]

From the relations (11) and (12), it follows that the function \( w \) as given by (10) satisfies the system

\[
\frac{d}{ds} \left( p \frac{dw}{ds} \right) + qw = AF(s) \quad (14)
\]

\[
w(a) = 0 \quad , \quad w'(a) = 0 . \quad (15)
\]

Dividing (14) by the constant \( A \) and comparing it with (5), we derive the required relation \( y(a) \) as

\[
y(s) = \int_a^s R(s,t)F(t)dt, \quad (16)
\]

Where

\[
R(s,t) = \left( \frac{1}{A} \right) \{u(s)v(t) - v(s)u(t)\}, \quad (17)
\]

Note that \( R(s,t) = -R(t,s) \).

It is easily verified that, for a fixed value of \( t \), the function \( R(s,t) \) is completely characterized as the solution of the initial value problem

\[
LR = \frac{d}{ds} \left( p(s) \frac{dR}{ds} \right) + q(s)R = \delta(s - t) ,
\]

\[
R|_{s=t} = 0 \quad , \quad \frac{dR}{ds}|_{s=t} = \frac{1}{p(t)} \quad (18)
\]

This function describes the influence on the value of \( y \) at \( s \) due to a concentrated disturbance at \( t \). It is called the influence function. The function \( G(s;t) \),

\[
G(s,t) = \begin{cases} 0 \quad , & s < t \\ R(s,t), & s > t \end{cases} \quad (19)
\]

Is called the causal Green's function.

Example. Consider the initial value problem.
\[ y'' + y = F(s), \quad 0 < s < 1, \quad y(0) = 0 \quad y'(0) = 0 \quad (20) \]

The influence function \( R(s, t) \) is the solution of the system

\[
\frac{d^2 R}{ds^2} + R = \delta(s-t), \quad R|_{s=t} = 0, \quad \frac{dR}{ds}|_{s=t} = 1. \quad (21)
\]

The required value of \( R \), clearly, is \( R(s, t) = \sin(s-t) \), and the integral representation formula for the initial value problem (20) is

\[
y(s) = \int_0^s \sin(s-t)F(t)dt \quad (22)
\]

When the value of \( y(a) \) and \( y'(a) \) are prescribed to be other than zero, then we simply add a suitable solution \( c_1u + c_2v \) of (7) to the integral equation (16) and evaluate the constants \( c_1 \) and \( c_2 \) by the prescribed conditions. For example,

\[
y'' + y = F(s), \quad y(0) = 1, \quad y'(0) = -1 \quad (23)
\]

Has the solution

\[
y(s) = \int_0^s [\sin(s-t)]f(t)dt + c_1(\sin s) + c_2(\cos s). \quad (24)
\]

With the help of the prescribed conditions, we find that \( c_1 = -1 \) and \( c_2 = 1 \).

### 3.5.2 Boundary value problems

Let us now consider the boundary value problem and start with the simplest one,

\[
\frac{d}{ds} \left( p \frac{dy}{ds} \right) + qy = F(s), \quad a \leq s \leq b, \quad (25)
\]

\[
y(a) = 0, \quad y(b) = 0 \quad (26)
\]

We attempt to write its general solution as an integral equation of the form

\[
y(s) = \int_a^s R(s, t)F(t)dt, + c_1u(s) + c_2v(s), \quad (27)
\]

Where \( u(s) \) and \( v(s) \) are the solution of the homogeneous equation (7). When we substitute the conditions (26) in (27), we obtain

\[
c_1 u(a) + c_2 v(a) = 0, \quad c_1 u(b) + c_2 v(b) = -\int R(b, t)F(t)dt,
\]
The function \( y(s) \) as given by (31) takes the elegant form

\[
y(s) = - \int G(s; t) F(t) dt
\]  

(34)

The function \( G(s; t) \) is called the Green's function. It is clearly symmetric.
Furthermore, it satisfies, for all t, the following auxiliary problem:

\[ LG = \frac{d}{ds} \left( p(s) \frac{dG}{ds} \right) + q(s)G = -\delta(s - t) \quad (36) \]

\[ G|_{a=a} = G|_{a=b} = 0 \quad (37) \]

\[ G|_{a=t+0} - G|_{a=t-0} = 0 \quad (38) \]

\[ \left. \frac{dG}{ds} \right|_{t+0} - \left. \frac{dG}{ds} \right|_{t-0} = -\frac{1}{p(t)} \quad (39) \]

Where by \( G|_{a=t+0} \) we mean the limit of \( G(s; t) \) as \( s \) approaches \( t \) from the right, and there are similar meanings for the other expressions. Thus, the condition (38) implies that the Green's function is continuous at \( s=t \). Similarly, the condition (39) states that \( \frac{dG}{ds} \) has a jump discontinuity of magnitude \(-\frac{1}{p(t)}\) at \( s=t \), the condition (38) (39) are called the matching conditions. It is instructive to note that the relation (39) is a consequence of the relations (35) and (36). Indeed, the value of the jump in the derivative of \( G(s; t) \) can be obtained by integrating (36) over small interval \((t-\varepsilon; s)\) and by recalling that the indefinite integral of \( \delta(s - t) \) is the Heaviside function \( H(s - t) \).

The result is

\[ p(t) \frac{dG(s; t)}{ds} + \int_{t-\varepsilon}^{s} q(x)G(x; t)dx = p(t-\varepsilon) \frac{dG(t-\varepsilon; s)}{ds} - H(s - t). \]

When \( s \) traverses the source point \( t \), then on the right side the Heaviside function has a unit jump discontinuity. Since other terms are continuous Function of \( s \), it follows that \( \frac{dG}{ds} \) has, at \( t \) a jump discontinuity as given by (39).

**Example.** Consider the boundary value problem

\[ y'' = F(s), \quad y(0) = y(\ell) = 0 \quad (40) \]

Comparing this system with the relations (25), (26), and (36)-(39), we readily evaluate the Green's function:
\[ G(s; t) = \begin{cases} \left( \frac{s}{\ell} \right) (\ell - t), & s < t, \\ \left( \frac{t}{\ell} \right) (\ell - s), & s > t \end{cases} \] 

(41)

Which is the kernel (5.3.10) except for the factor \( \lambda \). The solution of (40) now follows by substituting this value in the equation (34). Incidentally by introducing the notation

\[ s_\prec = \begin{cases} s, & s \leq t, \\ t, & s \geq t, \end{cases} \quad \text{or} \quad \min(s, t), \]

And

\[ s_\succ = \begin{cases} t, & s \leq t, \\ s, & s \geq t, \end{cases} \quad \text{or} \quad \max(s, t), \]

The relation (41) takes the compact form

\[ G(s; t) = \left( \frac{1}{\ell} \right) [s_\prec (\ell - s_\succ)], \quad 0 \leq s, t \leq \ell \] 

(42)

It follows from the properties (36)-(39) of the Green’s function \( G(s; t) \) that \( \partial G(s; a) / \partial t \) satisfies the system of equations

\[ \frac{d}{ds} \left[ p(s) \frac{d}{ds} \left( \frac{\partial G(s; a)}{\partial t} \right) \right] + q(s) \frac{\partial G(s; a)}{\partial t} = 0. \quad a < s < b \]

\[ \frac{\partial G(a; a)}{\partial t} = \frac{1}{p(a)}, \quad \frac{\partial G(b; a)}{\partial t} = 0 \] 

(43)

Similarly, \( \partial G(s; a) / \partial t \) satisfies the system

\[ \frac{d}{ds} \left[ p(s) \frac{d}{ds} \left( \frac{\partial G(s; a)}{\partial t} \right) \right] + q(s) \frac{\partial G(s; a)}{\partial t} = 0. \quad a < s < b \]

\[ \frac{\partial G(a; b)}{\partial t} = 0, \quad \frac{\partial G(b; b)}{\partial t} = -\frac{1}{p(b)} \] 

(44)
Hence, the boundary value problem

\[(py')' + qy = F, \quad y(a) = \alpha, \quad y(b) = \beta \quad (45)\]

Has the solution

\[y(s) = - \int G(s; t)F(t)dt + \alpha p(a) \frac{\partial G(s; a)}{\partial t} - \beta(b) \frac{\partial G(s; b)}{\partial t} \quad (46)\]

As is easily verified.

Finally, we present the integral-equation formulation for the boundary value problem with more general and inhomogeneous end conditions:

\[(py')' + qy = F(s), \quad - \mu_1 y'(a) + v_1 y(a) = \alpha \]
\[, \quad \mu_2 y'(b) + v_2 y(b) = \beta \quad (47)\]

When we proceed to solve this system in the same way as we did the system (1)-(2), the Green's function for the present case can also be derived provided the determinant

\[D = [-\mu_1 u'(a) + v_1 u(a)][\mu_2 v'(b) + v_2 v(b)] - \mu_1 v'(a) + v_1 v(a)[\mu_2 u'(b) + v_2 u(b)] \neq 0\]

Indeed, \(G(s; t)\) possesses the following properties:

\[LG = \frac{d}{ds} \left[ p(s) \frac{dG}{ds} \right] + q(s)G = - \delta(s-t) - \mu_1 \frac{dG|_{s=a}}{ds} + v_1 G|_{s=a} = \mu_2 \frac{dG|_{s=b}}{ds} + v_2 G|_{s=b} \]
\[G|_{s=t+0} - G|_{s=t-0} = 0 \]
\[\frac{dG}{ds}|_{s=t+0} - \frac{dG}{ds}|_{s=t-0} = - \frac{1}{p(t)} \quad (48)\]

And the condition of symmetry. With the help of the Green's function, the boundary value problem (47) has the unique solution

\[y(s) = \int G(s; t)F(t)dt + \frac{p(a)}{\mu_1} \alpha G(s; a) + \frac{p(b)}{\mu_2} \beta G(s; b) \quad (49)\]

Provided \(\mu_1\) and \(\mu_2\) do not vanish. If \(\mu_1 = 0\), then the factor \(\frac{1}{\mu_1} G(s; a)\) is replaced by \(1 / v_1 \partial G(s; a) / \partial t\). By the same token, if \(\mu_2 = 0\), we replace
When $\alpha$ and $\beta$ are zero, the relation (49) reduces to (34).

The Sturm-Liouville problem consists in solving a differential equation of the form

$$ (py')' + qy + \lambda ry = F(s), \quad (50) $$

Involving a parameter $\lambda$ and subject to a pair of homogeneous boundary conditions

$$ -\mu_1 y'(a) + v_1 y(a) = 0 \quad , \quad \mu_2 y'(b) + v_2 y(b) = 0 \quad (51) $$

The values of $\lambda$ for which this problem has a nontrivial solution are called the eigenvalues. The corresponding solutions are the Eigen-function. In case $p(a) = p(b)$, the boundary conditions (51) are replaced by the periodic boundary conditions

$$ y(a) = y(b) , \quad y'(a) = y'(b) $$

From formula (49), it follows that the solution of equation (50) subject to the conditions (51) is

$$ y(s) = \lambda \int r(t)G(s; t)y(t)dt - \int G(s; t)F(t)dt \quad (52) $$

Which is a Fredholm integral equation of the second kind.

In this equation, $G(s; t)r(t)$ is not symmetric unless the function $r$ is a constant. However, by setting

$$ [r(s)]^{\frac{1}{2}}y(s) = Y(s) , $$

Under the assumption that $r(s)$ is nonnegative over the interval $(a, b)$ the equation (52) takes the form

$$ y(s)[r(s)]^{\frac{1}{2}} = \lambda \int G(s; t)[r(s)]^{\frac{1}{2}}[r(t)]^{\frac{1}{2}}y(t)dt - \int G(s; t)[r(s)]^{\frac{1}{2}}[r(t)]^{\frac{1}{2}}F(t)\frac{1}{[r(t)]^{\frac{1}{2}}}dt , $$

Or

$$ Y(s) = \lambda \int \tilde{G}(s; t)Y(t)dt - \int \tilde{G}(s; t) \frac{F(t)}{[r(t)]^{\frac{1}{2}}} dt \quad (53) $$

Where $\tilde{G}(s; t) = G(s; t)[r(s)]^{\frac{1}{2}}[r(t)]^{\frac{1}{2}}$ is a symmetric kernel.
The above discussion on boundary value problem is based on the assumption that \( D = u(a)v(b) - v(a)u(b) \) does not vanish. If it vanishes, then the homogenous equations

\[ C_1 u(a) + C_2 v(a) = 0, \quad C_1 u(b) + C_2 v(b) = 0 \]

Have a nontrivial solution \((C_1, C_2)\), and the function \( w(s) = C_1 u(s) + C_2 v(s) \) satisfies the completely homogeneous system

\[
(pw')' + qw = 0, \quad w(a) = w(b) = 0 \quad (54)
\]

Therefore, if \( y \) is a solution of (25),(45), or(47), then so is \( y + cw \) for any constant \( c \). This means these systems do not have a unique solution. This is not all. There is an additional consistency condition which must be satisfied for these systems to have a solution. Take, for example, the system (45). Multiply the differential equation (45) by \( w \) and integrate from \( a \) to \( b \) and get

\[
\int w(s)F(s)ds = \int w(s)((py')' + qy)ds \\
= [wp'y - w'py]_a^b + \int y[(pw')' + qw)]ds = p(a)w'(a)\alpha - p(b)w'(b)\beta \quad (55)
\]

Therefore, it (45) is to have a solution, then the given function \( F(x) \) must satisfy the consistency condition (55). For \( \alpha = \beta = 0 \), we get the consistency condition

\[
\int w(s)F(s)ds = 0 \quad (56)
\]

For the system (25).

Thus, if \( D \) id zero, then we either have no solution or many solution; but never just one. , view [1]
3.6 EXAMPLE

3.6.1 Example 1  Reduce the boundary value problem

\[ y'' + \lambda y = 0, \quad (1) \]
\[ y(0) = 0, \quad y'(1) + v_2 y(1) = 1, \quad (2) \]

To a Fredholm integral equation.

From the properties (48)\textsubscript{1} and (48)\textsubscript{2}, we must have

\[ G(s; t) = \begin{cases} A_1(t)s, & s < t \\ A_2(t)[1 + v_2(1 - s)], & s > t \end{cases} \]

The consequence of the symmetry of the Green's function is

\[ A_2 = C[1 + v_2(1 - t)], \quad A_2 = Ct \]

Where C is a constant independent of t. The jump condition (48)\textsubscript{4} yields

\[ Ct(-v_2) - C[1 + v_2(1 - t)] = -1 \]

Or

\[ C = 1 / (1 + v_2). \]

Thus, the Green's function is completely determined:

\[ G(s; t) = \begin{cases} [1 + v_2(1 - t)]s/(1 + v_2), & s < t, \\ [1 + v_2(1 - s)]t/(1 + v_2), & s > t \end{cases} \]

(3)
References


